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A simple discrimination experiment with a continuum of responses

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This paper reports tests of several stochastic learning models for a continuum of responses in a simple discrimination experiment. The basic models are formulated in Suppes (1959, 1960). The only previous experiment in the literature directly testing these models, reported by Suppes and Frankmann (1961), is concerned with a simple learning situation with unimodal non-contingent determinate reinforcement. They describe their experiment as follows:

> The subject is told that his task on each trial is to predict by means of a pointer where a spot of light will appear on the circumference of a circle; the subject's responses are his pointer predictions. At the end of each trial the "correct" position of the spot is shown to the subject; this is the reinforcing event for the trial. The response x and the reinforcement y vary continuously along the circle from 0 to 2π.

The particular reinforcement distribution used by Suppes and Frankmann (1961) was the triangular distribution on the interval 0 to 2π.

The present experiment used the same circular apparatus. The most important modification was the introduction of a discrimination situation in the following manner: A panel of four lights was placed in front of the subject. At the beginning of each trial one of the four lights went on. Corresponding to each light was a different probability distribution for the reinforcement on the circle. The four reinforcement distributions were uniform with half-range l equal to $\pi/10$ radians for lights 1 and 3 and to $\pi/5$ radians for lights

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2 and 4. The four reinforcement regions were equally spaced on the circle with their relative positions identical for all subjects, although the absolute locations of the regions with respect to the horizontal and the vertical axes were randomized for different subjects.

In the previous experimental study all subjects were run for 300 trials. In the present study that number was increased to 500 trials in order that adequate data on each of the discriminative stimuli might be obtained.

The discrimination situation is an admittedly simple one, for there is no apparent overlap among the discriminative stimuli or among the four different reinforcement regions, and in fact each region is separated from the next by a significant zone of nonreinforcement. This simple situation was selected deliberately in order to facilitate detailed analysis of response data. Operating on the assumption that there is no overlap in the stimuli associated with the four lights, we are justified from a theoretical standpoint in analyzing the response data for each of the four discriminative lights as a separate learning process. In terms of stimulus-sampling theory this assumption amounts to postulating that there are exactly four stimuli (the four lights) and that exactly one of them is sampled on a given trial. (For a further discussion of this simplified approach to discrimination, see Suppes and Atkinson, 1960, pp. 139–140.)

The essential modification of the theory for a finite number of responses introduced in Suppes (1960) for handling a continuum of responses is the concept of a *smearing distribution*. In the finite case the fundamental assumption of stimulus-sampling theory is that each stimulus element is conditioned to at most one response; this assumption seems unsound in the case of a continuum. The introduction of a smearing distribution amounts to postulating that the conditioning of each stimulus is *smeared* over a certain interval of responses, possibly the whole continuum available. The conditioning of any stimulus is represented by a probability density $k(x; y)$ for the response x. The parameter y is the mean of the k distribution. It is postulated that y takes on the value of the point of reinforcement when the stimulus is sampled and reinforcement is effective. The conditioning parameter c is the probability of reinforcement being effective on any trial.

The smearing distribution enters into the linear models in a similar fashion, and the learning parameter θ plays essentially the same role as the conditioning parameter c. Because the predictions we consider are identical for the stimulus-sampling and linear models, we shall mainly consider linear models in the sequel, in view of their greater simplicity of formulation. However, from the standpoint of psychological theory, we consider the stimulus-sampling models to be more fundamental because they postulate schemata for the processes of conditioning and sampling stimuli; for a detailed statement of their basic assumptions the reader is referred to Suppes (1960).

The exact shape assumed for the smearing distribution had little effect on the predicted response distributions in the Suppes and Frankmann (1961) study. The uniform reinforcement distributions used in the present experiment provide a better opportunity to discriminate between various smearing distributions. It is to be emphasized that the smearing distribution is a theoretical concept and must be inferred from the response histogram. It is not possible to make direct frequency observations on the smearing distribution itself. Finally, before we turn to a presentation of the theoretical results tested in the experiment, it should be mentioned that we also present at a grosser level an analysis of the association formed between the discriminative stimuli, i.e., the four lights, and the regions of reinforcement. This analysis is performed in terms of the one-stimulus-element, two-response model that has been much applied to paired-associate and simple concept learning. A brief characterization of the model is given at the beginning of the discussion of the association process.

1. Summary of theoretical results

We will not give here a general development of the linear models or of the stimulus-sampling models for a continuum of responses but will present only basic ideas and results actually used in the analysis of data. Detailed proofs and derivations are given in the Appendix. We also restrict ourselves to the case where for each discriminating stimulus the reinforcement schedule is *determinate*, i.e., a reinforcement follows a response at each trial; and *noncontingent*, i.e., independent of the subject's responses. In this section we first discuss the general concepts common to the different models; then after recalling the main features of the basic model (Model I), we investigate various extensions of it. We conclude with some remarks on the estimation of the smearing distribution.

The first concept to be discussed is that of the reinforcement distribution. Letting y be the point of reinforcement, we denote by $f(y)$ the *reinforcement density*. Having in mind the particular (uniform) distribution used in the experiment, we assume $f(y)$ to be symmetric around the origin, and of range $2l$, i.e., $f(y) > 0$ for $-l \leq y \leq l$, and $f(y) = 0$ for $|y| > l$. Given an interval $Y = (a, b)$, we denote by $F(Y)$ the integral of $f(y)$ over Y, i.e., the probability of a reinforcement inside the interval Y.

Although the reinforcement density $f(y)$ is known, the smearing distribution is a hypothetical construct. It can be characterized by a function $k(x, y)$, which, when considered as a function of x, is a probability density, symmetric around the point $x = y$. In this study, we also assume that $k(x, y)$ has finite range $2a$ and that a is not too large; in the case of the circle, if we suppose $a < \pi - l$, we can apply to the circle the formulas valid for a linear continuum because there is no periodicity in the functions f and k. As we shall see, the data support this restriction on a.

We use x_n for the response on trial n and $j_n(y_n x_n s_{n-1})$ for the joint density on trial n, where s_{n-1} is the finite sequence of responses and reinforcements from trial 1 through trial $n-1$. The mean asymptotic response density we denote by $r(x)$, applying subscripts I, II, and III, as appropriate, for the various models we consider. Using the preceding concepts, we consider different models: Model I is the basic model already developed in Suppes (1959). Models II, II′, and III are extensions of Model I. For each of these extensions, we might consider a linear model and a stimulus-sampling model. Both lead to predictions that agree in certain cases but differ for some sequential statistics. In fact, in the present experiment, the theoretical quantities actually checked against the data coincide for both kinds of models; therefore we will restrict ourselves, in the following presentation, to the linear models.[1]

The basic model (Model I). We first summarize the basic properties of Model I. Most proofs are given in Suppes (1959). The basic axiom can be written considering densities instead of distribution functions:

$$j_n(x \mid y_{n-1}, x_{n-1}, s_{n-2}) = (1 - \theta) j_{n-1}(x \mid s_{n-2}) + \theta k(x, y_{n-1}). \tag{1}$$

The basic feature of this model is that the center of the smearing density is y_{n-1}, i.e., the point of the last reinforcement.

The derivation from Eq. (1) of the asymptotic response density, averaged over all possible past histories, yields the following equation:

$$r_I(x) = \int_{-l}^{+l} k(x, y) f(y)\, dy, \tag{2}$$

where $f(y)$ is the noncontingent reinforcement density; the asymptotic probability of a response inside any given interval X is given by

$$R_I(X) = \int_X r_I(x)\, dx,$$

a notation we use repeatedly.

The joint asymptotic density of the response x_n following the reinforcement y_{n-1} can be shown to be

$$j(x_n, y_{n-1}) = (1 - \theta) r_n(x_n) f(y_{n-1}) + \theta f(y_{n-1}) k(x_n, y_{n-1}).$$

Now taking intervals X, Y, we can obtain the asymptotic conditional probability of a response in the interval X_n given the preceding reinforcement in the interval Y_{n-1}:

$$P(X_n \mid Y_{n-1}) = (1 - \theta) R_I(X_n) + \theta \frac{H(X_n, Y_{n-1})}{F(Y_{n-1})}, \tag{3}$$

where

$$H(X, Y) = \int_X \int_Y k(x, y) f(y)\, dx dy,$$

and similarly, we can obtain the conditional probability of a response in X_n

[1] See the Appendix for some corresponding stimulus-sampling models.

given the two preceding reinforcements in Y_{n-1} and Y_{n-2}:

$$(4)\qquad P(X_n \mid Y_{n-1}, Y_{n-2}) = (1-\theta)^2 R_I(X_n) + \theta\,\frac{H(X_n, Y_{n-1})}{F(Y_{n-1})} + \theta(1-\theta)\,\frac{H(X_n, Y_{n-2})}{F(Y_{n-2})}.$$

Models II, II′, and III. As will be shown in the section on empirical results, the fit of Model I is not satisfactory for the two large reinforcement regions with half-range equal to $\pi/5$ radians. A plausible hypothesis to account for this discrepancy is that the strict independence-of-path assumption implied by Eq. (1) is violated. Subjects may, for instance, tend to average the most recent reinforcements, perhaps using some weighting function, and may not simply be affected in changing their response distributions by the immediately preceding reinforcement. When the reinforcement distribution is uniform over a restricted interval of the continuum of possible responses it seems particularly natural to expect some piling up of responses around the center of this distribution. To test this kind of hypothesis we have developed several modifications of Model I that take such dependency possibilities into account. For obvious reasons we have restricted ourselves to three models that are computationally manageable.

These models retain the structure of the recursion axiom (1), but we rewrite this axiom as follows:

$$(5)\qquad j_n(x \mid y_{n-1}, x_{n-1}, s_{n-2}) = (1-\theta) j_{n-1}(x \mid s_{n-2}) + \theta k(x, z_n),$$

where z_n, the center of the smearing distribution, is not necessarily y_{n-1}, the point of the last reinforcement.

Within this framework there is still considerable freedom, even if we restrict z_n to be, for instance, a weighted combination of previous reinforcements. If we take $z_n = y_{n-1}$, we get Model I. Another extreme possibility is to take $z_n = A_n$, where A_n is the over-all average of all previous reinforcements. In the noncontingent case, A_n converges to a fixed point, the mean of the reinforcement distribution (which we can take equal to 0); in this case the asymptotic response distribution would be just the smearing distribution with its center at 0.

In this paper we mainly investigate two intermediate cases, which we call Model II and Model III. Model II is defined by letting $z_n = (y_{n-1} + y_{n-2})/2$; i.e., z_n is the mean of the two previous reinforcements. Model III is defined by letting $z_n = (y_{n-1} + 0)/2 = y_{n-1}/2$. In this case z_n is at asymptote the mean of the last reinforcement y_{n-1} and of the over-all average of reinforcements, 0. Stated formally, the new recursion axioms corresponding to Eq. (1), read:

Model II:

$$(6)\qquad j_n(x \mid y_{n-1}, x_{n-1}, y_{n-2}, x_{n-2}, s_{n-3}) = (1-\theta) j_{n-1}(x \mid y_{n-2}, x_{n-2}, s_{n-3}) + \theta k\left(x, \frac{y_{n-1} + y_{n-2}}{2}\right);$$

MODEL III:

$$j_n(x \mid y_{n-1}, x_{n-1}, s_{n-2}) = (1-\theta)j_{n-1}(x \mid s_{n-2}) + \theta k\left(x, \frac{y_{n-1}}{2}\right). \tag{7}$$

In addition to Models II and III, which correspond to the structure described by Eq. (5), we have studied a model with a recursion axiom going back two trials instead of one, and leading to predictions close to those of Model II; we call it

MODEL II′:

$$\begin{aligned} &j_n(x \mid y_{n-1}, x_{n-1}, y_{n-2}, x_{n-2}, s_{n-3}) \\ &\qquad = (1-\theta)j_{n-2}(x \mid s_{n-3}) + \theta k\left(x, \frac{y_{n-1}+y_{n-2}}{2}\right). \end{aligned} \tag{8}$$

Its intuitive idea is essentially the same as that of Model II, but it decreases the impact of the reinforcement on trial $n-2$ on the response on trial n.

We now indicate for each model the asymptotic response density and the two sequential probabilities corresponding to those given for Model I, i.e., the analogues of Eqs. (2), (3), and (4).

For the asymptotic response densities, we have

MODELS II AND II′:

$$r_{\text{II}}(x) = 2\int_{-l}^{+l}\int_{-l}^{+l} f(u)f(2y-u)k(x,y)\,dy du\,; \tag{9}$$

MODEL III:

$$r_{\text{III}}(x) = 2\int_{-l}^{+l} f(2y)k(x,y)\,dy. \tag{10}$$

Notice that Eq. (9) can be rewritten in a way formally identical to Eq. (2). If we let

$$f'(y) = 2\int_{-l}^{+l} f(u)f(2y-u)\,du,$$

Eq. (9) becomes

$$r_{\text{II}}(x) = \int_{-l}^{+l} f'(y)k(x,y)\,dy.$$

The function $f'(y)$ is a density, corresponding to the distribution of the average of two variables distributed according to the density $f(y)$; thus one sees that Model II yields the asymptotic response distributions that Model I would yield if the reinforcement density $f(y)$ were replaced by the new reinforcement density $f'(y)$. For that reason, $f'(y)$ may be called a "pseudo-reinforcement density." Similarly, for Model III, a pseudoreinforcement density can be defined as $f''(y) = 2f(2y)$. It should be pointed out, however, that by this change Models II and III cannot be reduced to Model I, as will be seen by considering the sequential statistics.

We consider first the analogues of Eq. (3), i.e., the probability of a response in X_n given the preceding reinforcement in Y_n:

MODELS II AND II′:

$$P(X_n \mid Y_{n-1}) = (1-\theta)R_{\mathrm{II}}(X_n) + \theta\frac{H'(X_n, Y_{n-1})}{F(Y_{n-1})}, \tag{11}$$

where

$$H'(X, Y) = \int_X\int_Y\left[\int_{-l}^{+l} k\left(x, \frac{y+u}{2}\right) f(u)\, du\right] f(y)\, dxdy.$$

MODEL III:

$$P(X_n \mid Y_{n-1}) = (1-\theta)R_{\mathrm{III}}(X_n) + \theta\,\frac{H''(X_n, Y_{n-1})}{F(Y_{n-1})}, \tag{12}$$

where

$$H''(X, Y) = \int_X\int_Y f(y)k\left(x, \frac{y}{2}\right) dxdy.$$

We next give the analogues of Eq. (4), i.e., the probability of a response in X_n given the two preceding reinforcements. Here (for the first time) Models II and II′ differ:

MODEL II:

$$P(X_n \mid Y_{n-1}, Y_{n-2}) = (1-\theta)^2 R_{\mathrm{II}}(X_n) + \theta\,\frac{L(X_n, Y_{n-1}, Y_{n-2})}{F(Y_{n-1})F(Y_{n-2})}, + (1-\theta)\theta\,\frac{H'(X_n, Y_{n-1})}{F(Y_{n-1})} \tag{13}$$

where

$$L(X, Y, Y') = \int_X\int_Y\int_{Y'} k\left(x, \frac{y+y'}{2}\right) f(y)f(y')\, dxdydy'.$$

MODEL II′:

$$P(X_n \mid Y_{n-1}, Y_{n-2}) = (1-\theta)R_{\mathrm{II}}(X_n) + \theta\,\frac{L(X_n, Y_{n-1}, Y_{n-2})}{F(Y_{n-1})F(Y_{n-2})}. \tag{14}$$

MODEL III:

$$P(X_n \mid Y_{n-1}, Y_{n-2}) = (1-\theta)^2 R_{\mathrm{III}}(X_n) + \theta\,\frac{H''(X_n, Y_{n-1})}{F(Y_{n-1})} + \theta(1-\theta)\,\frac{H''(X_n, Y_{n-2})}{F(Y_{n-2})}. \tag{15}$$

Estimation of the smearing distribution. The smearing distribution must be estimated from the data. Theoretically, Eq. (1) [or (9) or (10), according to the model being considered] completely determines the smearing distribution from the knowledge of the asymptotic response distribution $r(x)$. Previous results, however, suggest that the asymptotic response distribution is not particularly sensitive to large differences in the form of the smearing distributions (Suppes and Frankmann, 1961). We will thus follow the procedure of assuming different types of smearing distributions (such as uniform or beta distributions) and predicting for each type the theoretical response distribution. However, it is valuable to possess a method of estimating important characteristics of the smearing distribution, such as the variance, before having to assume the distribution to be of any particular type. For the present experiment, such a method is a consequence of the following property.

Additive variance property. For a linear continuum the variance of the asymptotic response distribution is the sum of the variance of the reinforcement (or pseudoreinforcement) distribution and the variance of the smearing distribution. We have used this variance property to make a first comparison of the different models on the basis of individual data.

2. Experimental method

Subjects. The subjects were 31 male and 13 female students at Stanford University. Each subject was paid four dollars for participating in the two-hour experimental session.

Apparatus. The general apparatus is the one described in Suppes and Frankmann (1961) but of the two circles therein described, only the larger (5 feet in diameter) was used in the present study. In addition, a square panel of four lights was introduced in front of the subject on the arm of his chair. The colors of the lights were green, yellow, blue, and red. One light went on at the beginning of each trial.

Procedure. The essential part of the instructions read to the subjects was as follows:

> In this experiment we are studying how people learn to locate targets on radar screens and how this skill can be developed. The experiment consists of a series of trials. On each trial there is a target located at some point on this screen, and your task is to try to predict as accurately as possible the location of that target. On each trial there is a different target.
>
> Now for the details. Do you see the bar of light on the screen? That bar of light can be rotated by turning this knob. Try it. You will find that the light can be moved around a large circle. On each trial the target will be a *point* located somewhere on the edge of this circle.
>
> Now look at the four lights you have on the arm of your chair: red, blue, green, and yellow (the order in which they were named was randomized). These are signal lights; they announce the beginning of each trial. When any one light goes on, make your prediction. Take the knob and turn it to move the bar of light to the point where you think the target lies. When the bar is at that point, release the knob and say "Mark." After you have made your prediction, you will be shown the location of the target. The bar of light will move from your prediction point to the correct position of the target, and I will say "Target." That is the end of a trial. The next trial begins when any one of the signal lights goes on again. Then you take the knob again and turn it to predict the location of a new target.

I will briefly repeat the steps of each trial:

1. Wait until any one of the signal lights goes on.
2. Take the knob, move the bar of light to your prediction point, release the knob, and say "Mark."
3. The bar of light moves toward the actual location of the target and I say "Target."

Remember that your task is to make as accurate a prediction as possible on each trial. Of course, you will have to guess on the first trials, but with practice you will see that your predictions are improving.

One last comment: go as quickly as possible.

As soon as the questions had been answered by paraphrasing the instructions, 500 trials were run, with two interruptions of about 6 minutes each after the 200th trial and the 350th trial. The average rate was 5 trials per minute.

Design. All 44 subjects were run under the same experimental conditions. The 500-trial reinforcement sequences were computed on the basis of the uniform distributions for the four regions as described in the introduction. The order of presentation of the four discriminating lights was random with one restriction: no light was used twice in succession. The physical location of the zero point of the scale on the circumference of the circle was chosen randomly for each subject.

3. Results and discussion

We have organized the experimental results under the following main headings: comparison of reinforcement regions of the same size; the association process; learning curves for variance; individual asymptotic data; asymptotic response distribution; and sequential statistics.

Before we turn to details, a general remark about the goodness-of-fit tests we apply is in order. We use χ^2 tests, but the usual criterion is not completely justified because the observations we consider are usually not independent, although specific measures to guarantee approximate independence have been taken in some of the cases reported below. Two remarks are to be made about this lack of independence. When independence is lacking, the observations are usually positively correlated, which may tend to increase systematically the χ^2 values. On the other hand, for most of the analyses given below, the number of observations is so large that the absence of independence is probably not a major factor affecting the value of the χ^2. What is at least as important to observe is that with the large number of observations used for the tests, the application of a conventional significance level is not too meaningful because the errors of the second kind have been reduced essentially to zero. To apply χ^2 tests in which errors of the first and second kind have been equated would be more satisfactory. To apply the non-central χ^2 distribution in this fashion requires that a clearly defined alternative hypothesis be stated, and although this is feasible for some of the results reported below we have not made the additional necessary computations. From a scientific standpoint we believe that a valuable comparison of the various models considered can be adequately inferred from the relative value of the different χ^2's, and this is the simplified procedure we have adopted.

Comparison of regions of same size. From the way in which the experiment was designed, with two pairs of identical reinforcement regions symmetrically placed on the circumference of the circle, a similarity of the results within each pair was to be expected. The examination of the data, including analysis of individual data, confirmed that expectation. Figure 1 shows the response variances of the four regions for the entire group of subjects. The close similarity of the variance curves for the two small regions (1 and 3) and the two large regions (2 and 4) is evident from the figure. Table 1 presents the detailed response histograms of the four regions for the last 300 trials. The close similarity between the two small regions on the one hand and the two large regions on the other is also apparent from inspection

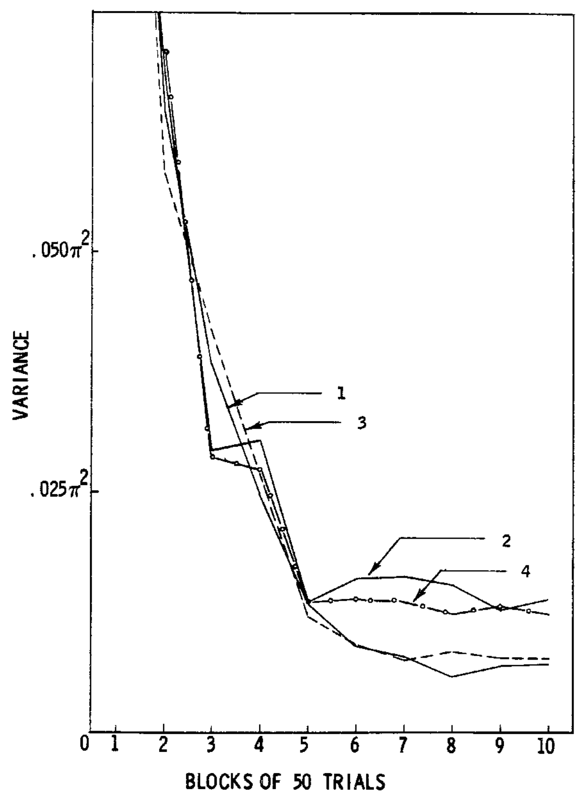

FIG. 1. Observed variances of responses in blocks of 50 trials for all four regions.

TABLE 1

RESPONSE HISTOGRAMS FOR THE LAST 300 TRIALS FOR THE FOUR REGIONS AND THE POOLED REGIONS S, S′, L, and L′

(Twenty-two classes of amplitude $.02\pi$ and two tail classes, 1 and 24)

Class	Regions and Pooled Regions							
	1	3	S	S′	2	4	L	L′
1	34	30	64	65	48	46	94	106
2	13	11	24	21	17	30	47	55
3	18	11	29	35	51	46	97	128
4	20	16	36	43	104	74	178	211
5	42	49	91	79	140	128	268	262
6	65	62	127	114	114	117	231	273
7	111	107	218	218	145	144	289	308
8	186	202	388	365	184	151	335	392
9	264	233	497	515	179	191	370	386
10	281	266	547	595	204	203	407	409
11	283	278	561	608	184	182	366	376
12	307	338	645	618	201	231	432	409
13	311	311	622	649	234	208	442	465
14	319	325	644	597	215	192	407	397
15	277	314	591	543	229	205	434	432
16	240	251	491	473	215	207	422	406
17	210	179	389	412	194	208	402	345
18	121	107	228	228	166	163	329	310
19	81	49	130	143	119	159	278	236
20	38	37	75	87	118	122	240	246
21	25	23	48	41	123	107	230	197
22	25	17	42	36	64	77	141	110
23	13	8	21	24	33	38	71	63
24	26	31	57	56	67	58	125	113
Total	3310	3255	6565	6565	3348	3287	6635	6635

of this table. For a more detailed comparison of the data in Table 1, it may be observed first that there are two different ways to compare the two small regions or the two large regions. For two regions of the same kind the observations can be counted in the same direction, for example clockwise for both regions, or in opposite directions, for example clockwise for region 1 and counter-clockwise for region 3. We first report the results of χ^2 tests of homogeneity of the data from the two regions of the same kind when the observations are counted in the same direction. These tests were performed on every other observation in order to approximate independence more closely than is possible with the successive observations. Classes of small frequency (less than 10 observations) were pooled, and as a result there were 20 class intervals in each histogram.[2] For the small regions 1 and 3, $\chi^2 = 16.3$, which for 20 classes and thus 19 df is not significant. For the large regions 2 and 4, $\chi^2 = 20.8$, which for the same number of classes is also not significant. When opposite directions are used for combining the two regions of a given kind for the two small regions, $\chi^2 = 17.7$, and for the two large regions $\chi^2 = 39.4$. The last χ^2 for the large regions is statistically significant at the .01 level. It seems very likely that the explanation of this significance is that subjects followed a clockwise stereotype by turning a knob in the same direction, but this discrepancy is not large and will not be further analyzed. Its existence, however, is one reason for adding to the subsequent analysis data for the regions combined in opposite directions, because this combination tends to eliminate the clockwise stereotype and to produce more symmetrical response distributions for the combined small region or combined large region.

The combination of regions 1 and 3 taken in the same direction is designated region S, and the combination of regions 2 and 4 taken in the same direction is called region L. When the regions are combined by taking them in opposite directions, we use primes and designate the regions S′ and L′, respectively. In view of the close identity of the responses for regions 1 and 3 on the one hand and regions 2 and 4 on the other, most of the subsequent analysis will be in terms of the combined regions S and L.

Association process. Before analyzing the experimental data in detail in terms of the continuum of responses, it is appropriate to consider the data from a grosser standpoint in terms of the subject's establishing an association between each of the four reinforcement regions of the circle and the corresponding discriminating light. For the purposes of this gross analysis we apply the all-or-none conditioning model with two responses, which has been successfully applied in recent years to paired-associate and concept-formation experiments (Bower, 1961; Estes, 1961; Suppes and Ginsberg, 1961, 1962). The essential nature of this model may be described in very simple terms. It is postulated that there is a single stimulus element. Until this single element is conditioned, i.e., until the association between the stimulus and response is established, there is a constant guessing probability p of a correct response. When a stimulus element is conditioned, the correct response is made with probability 1. Secondly, on each trial before conditioning there is a constant probability c that the single stimulus element will be conditioned to the correct response.

To perform a statistical analysis of the data from the standpoint of this all-or-none model, it is necessary to choose a criterion of "complete" learning. This choice is somewhat arbitrary because some subjects gave occasional responses outside the appropriate reinforcement region during the entire experiment. Also, the definition of a correct response creates a problem, for if

[2] Because the data of Table 1 are based on every observation, this table does not show which classes had frequency less than 10.

we accept as correct only those responses that fall inside the reinforcement region, then we have too strict a definition for the small regions, and most subjects meet no reasonable criterion of learning. Solving the two problems jointly, we have chosen for each kind of region, large and small, two different definitions of the correct response and of the criterion for learning. For region S with the reinforcement interval of length $.20\pi$, we have analyzed the data according to the following two definitions. In definition S_1, we require the response to be in the interval length of $.24\pi$ and the learning criterion to be three successive correct responses. The alternative definition S_2 is more liberal, permitting the response interval to have a length of $.32\pi$ and the learning criterion again to be three successive correct responses.

For the large region, definition L_1 requires the response interval to be of length $.44\pi$ and the learning criterion to be four successive correct responses. The alternative definition L_2 requires the response interval to be of length $.40\pi$ and the learning criterion to be three successive correct responses. It should be noted that the criterion of three or four successive correct responses is fairly stringent. For it is reasonable to assume that before the association is established, subjects have a roughly uniform response distribution on the circle and thus even the probability of making three correct responses with criterion L_2 is only $(.40\pi/2\pi)^3 = .008$.

Using these criteria, we analyze the results in terms of the statistics introduced in Suppes and Ginsberg (1961); for details of the statistical tests see their article. The most important observation for the application of their statistics is that if the response data are restricted to responses prior to the last error, then what should be obtained is a sequence of Bernoulli trials with constant binomial parameter p.

On this assumption there should be no increase in proportion of correct responses prior to the last error (where of course the last error is defined in terms of the criteria stated above). A very sensitive test of this null hypothesis of the all-or-none model concerning stationarity for responses prior to the last error is provided by the construction of Vincent pre-criterion learning curves. Curves for the two regions and for the two criteria of each region are shown in Fig. 2 in terms of the proportion of errors in each quartile prior to the last error. The proportion for each quartile is obtained by summing the number of correct responses for each subject in that quartile of his responses and then dividing the total correct responses by the total number of responses for all subjects. The curves in Fig. 2 indicate that the null hypothesis of stationarity is approximately confirmed, although there is a slight tendency for the proportion of errors to decrease as the criterion is approached. The corresponding χ^2 tests for stationarity are as follows: $\chi^2(S_1) = 3.67$, $\chi^2(S_2) = 2.46$, $\chi^2(L_1) = 1.75$, and $\chi^2(L_2) = .17$. With three degrees of freedom, none of these chi-squares are significant.

Similar stationarity results are obtained from a trial-by-trial analysis. In Table 2, χ^2 tests for stationarity, order, and binomial distribution are shown

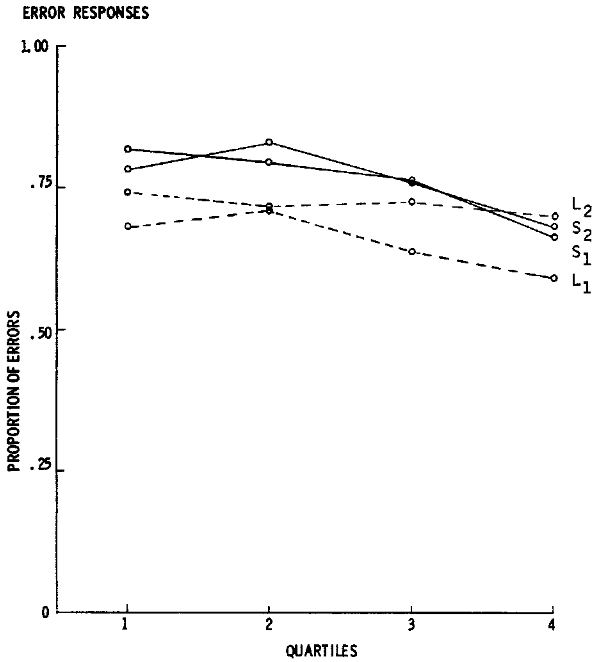

FIG. 2. Vincent learning curves in quartiles for proportion of errors prior to last error for both regions S and L under two different criteria of learning.

for S_1, S_2, L_1, and L_2. The stationarity tests are for blocks of four trials, and again the null hypothesis is that prior to the last error, the learning curve is horizontal. The null hypothesis of the order tests is that the responses prior to the last error are independent. The χ^2 test used here is for the hypothesis of independence vs. that of first-order dependence. Finally, the test for the binomial distribution is in terms of blocks of four trials. More exactly, the binomial-distribution test is constructed in the following fashion. We con-

TABLE 2

RESULTS OF CHI-SQUARE TESTS OF STATIONARITY, ORDER, AND BINOMIAL DISTRIBUTION FOR ASSOCIATION PROCESS

(Degrees of freedom are shown in parentheses)

Test	Region			
	S_1	S_2	L_1	L_2
Stationarity	3.06 (10)	3.12 (9)	1.84 (8)	1.05 (6)
Order	0.05 (1)	1.27 (1)	0.02 (1)	3.65 (1)
Binomial distr.	0.76 (1)	0.11 (1)	1.14 (2)	0.25 (1)

TABLE 3

OBSERVED AND PREDICTED FREQUENCY DISTRIBUTIONS OF SEQUENCES OF ERRORS AND SUCCESSES OVER BLOCKS OF FOUR TRIALS, POOLED REGIONS S_1, S_2, L_1, AND L_2, WITH 1 = SUCCESS, 0 = ERROR. ESTIMATED $\hat{p}$ and χ^2 RESULTS SHOWN

Response Sequence	Region							
	$S_1(\hat{p} = .230)$		$S_2(\hat{p} = .219)$		$L_1(\hat{p} = .332)$		$L_2(\hat{p} = .279)$	
	Obt. Freq.	Pred. Freq.	Obt. Freq.	Pred. Freq.	Obt. Freq.	Pred. Freq.	Obt. Freq.	Pred. Freq.
0000	73	69.99	57	59.60	31	28.74	32	32.40
1000	20	20.89	17	16.69	7	14.26	9	12.55
0100	21	20.89	20	16.69	13	14.26	9	12.55
0010	20	20.89	20	16.69	14	14.26	12	12.55
0001	17	20.89	12	16.69	20	14.26	18	12.55
1100	5	6.24	4	4.67	5	7.07	3	4.86
1010	8	6.24	6	4.67	8	7.07	7	4.86
1001	7	6.24	4	4.67	4	7.07	4	4.86
0110	8	6.24	6	4.67	12	7.07	11	4.86
0101	8	6.24	8	4.67	9	7.07	8	4.86
0011	3	6.24	3	4.67	2	7.07	1	4.86
1110	0	1.86	0	1.31	4	3.51	0	1.88
1101	3	1.86	0	1.31	3	3.51	3	1.88
1011	6	1.86	3	1.31	8	3.51	3	1.88
0111	0	1.86	0	1.31	4	3.51	0	1.88
1111	0	0.56	0	0.37	0	1.74	0	0.73
Total	199	198.99	160	159.99	144	143.98	120	120.01
χ^2	0.97 , 6 df		2.74 , 6 df		7.99 , 6 df		7.85 , 6 df	

sider blocks of trials of length 4 and for each subject use the highest multiple of a block equal to or less than the total number of responses prior to the last error. Summing over subjects, we construct the histogram of the response distribution for this block length. The number of degrees of freedom for each of the χ^2's in Table 2 is indicated in the subscript; none of the values obtained approaches significance.

In addition to considering the distribution of responses, we can analyze the data in a still more detailed way by considering the distribution of sequences of responses. Distributions of the 16 sequences of responses for blocks of four trials are shown in Table 3. If we consider the relative frequency of each sequence of responses of length 4, a χ^2 test may be applied. It is worth noting that the goodness of fit of the distribution of these sequences provides a test for the kind of run statistics much studied in the literature of learning theory. The χ^2 values and the indicated degrees of freedom shown at the bottom of

Table 3 indicate that for none of the four cases is there a significant deviation from the null hypothesis of a binomial distribution. The number of degrees of freedom is reduced because of the small number of observations in some cells. The results of these various tests support the hypothesis that the association process in the present experiment was established on an approximate all-or-none basis. In this respect the results may be regarded as supporting and extending those obtained in paired-associate and concept-learning experiments.

Learning curves for variance. The evolution of the group response characteristics over trials is summarized in Table 4, which gives means and variances of regions S and L for blocks of 50 trials. For each region, the midpoint is set equal to zero. The closeness of the means to zero shows that the symmetry of the reinforcement distribution for each region is rapidly reflected in the response distribution.

Plotting the variances against trials, we get the empirical curves of Fig. 3.

In the Suppes and Frankmann study (1961) where Model I was used, an estimate of θ was obtained from the rate of decrease of the experimental response variance σ_n^2. In fact, it can be shown that the recursion formula

$$r_{n+1}(x) = (1 - \theta)r_n(x) + \theta r(x)$$

as well as the approximate recursion formula

$$\sigma_{n+1}^2 = (1 - \theta)^2\sigma_n^2 + \theta\sigma^2$$

(where σ^2 is the asymptotic variance) holds for all models investigated in this paper. Therefore, the latter formula provides an over-all estimate of θ, valid for all models. By using the method already used in the Suppes and Frank-

TABLE 4

OBSERVED MEANS AND VARIANCES OF RESPONSES OVER ALL TRIALS IN BLOCKS OF 50 FOR POOLED REGIONS S AND L

(Means have been divided by π and variances by π^2)

Blocks	Region S		Region L	
	Mean	Variance	Mean	Variance
1	.0009	.15417	.0134	.14694
2	.0021	.06156	.0003	.06945
3	.0030	.04026	.0002	.02899
4	.0013	.02604	.0086	.02893
5	.0010	.01267	.0038	.01357
6	.0013	.00904	.0029	.01509
7	−.0007	.00770	.0049	.01496
8	−.0035	.00700	.0039	.01388
9	−.0036	.00733	.0099	.01296
10	−.0036	.00746	.0021	.01320

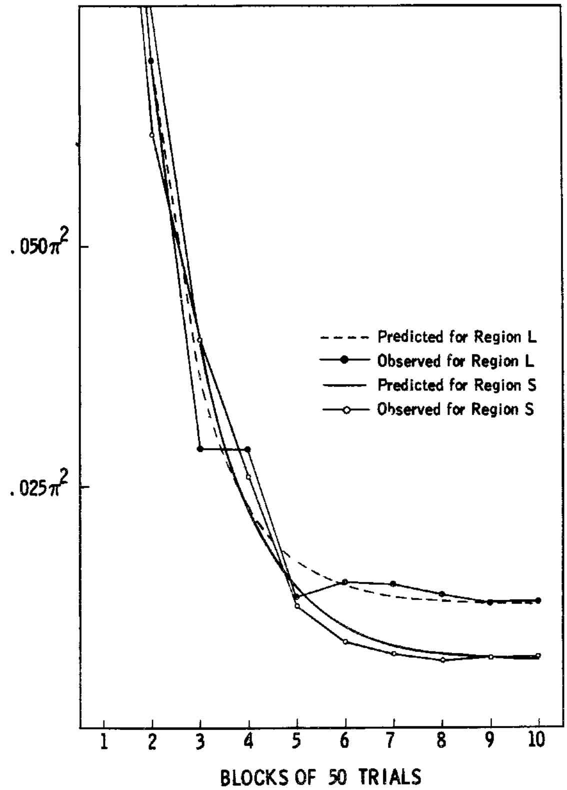

FIG. 3. Observed variances of responses in blocks of 50 trials and predicted curves for regions S and L.

mann study (1961) and averaging over blocks of 50 trials, we fitted the experimental curve by a least-squares method and obtained the following estimates of θ:

$$\text{For region S, } \theta^* = .015\,,$$
$$\text{For region L, } \theta^* = .017\,.$$

These values are notably smaller than those obtained in the Suppes and Frankmann (1961) study (.065 and .033). The corresponding theoretical curves are plotted in Fig. 3 together with the empirical curves.

Some different methods of estimating θ are considered below.

Individual asymptotic data. This section, as well as the two immediately following, is devoted to the asymptotic predictions of the various models. The corresponding asymptotic data will be defined as the data relative to the last 300 trials. The results shown in Fig. 3 justify the choice of the cutoff point, which also coincides with the first interruption in the experiment.

Starting with individual data, we computed the means, variances, and

TABLE 5

INDIVIDUAL OBSERVED MEANS, VARIANCES, AND THIRD MOMENTS FOR LAST 300 TRIALS FOR POOLED REGIONS S AND L

(Means have been divided by π, variances by π^2, and third moments by π^3)

Subject	Region S			Region L		
	Mean	Variance	Third Moment	Mean	Variance	Third Moment
1	.0126	.00399	.00010	.0073	.01142[a]	−.00000
2	−.0033	.00983	.00001	−.0045	.01358	.00006
3	.0028	.00569	.00036	−.0176	.01607	.00078
4	−.0390	.00684	−.00001	−.0004	.08175[a]	.00015
5	.0123	.00468	.00064	−.0001	.01213[a]	.00121
6	−.0101	.00861	.00028	.0280	.01766	.00182
7	.0115	.00611	.00067	.0055	.00792[a]	−.00013
8	.0019	.00797	.00017	.0193	.01146[a]	.00114
9	.0030	.00285[a]	.00058	.0010	.00550[b]	−.00091
10	.0229	.00454	.00073	.0573	.01014[a]	−.00036
11	−.0084	.00355	.00006	.0575	.01389	−.00021
12	.0264	.01427	.00317	.0251	.01061	.00028
13	−.0115	.01587	.00047	−.0209	.01739	.00513
14	−.0237	.00295[a]	.00012	.0065	.01241[a]	−.00050
15	−.0228	.00883	−.00010	.0001	.00570[b]	−.00000
16	−.0120	.00224[a]	.00002	.0101	.01517	.00586
17	−.0012	.00432	−.00003	.0001	.01769	.00007
18	−.0091	.00662	−.00048	.0073	.01397	−.00011
19	−.0063	.00370	−.00027	.0081	.02178	−.00594
20	.0163	.02933	.01914	.0111	.02903	−.00301
21	−.0410	.00872	−.00019	−.0158	.01818	.00052
22	.0268	.00770	.00024	.0403	.00638[b]	−.00006
23	−.0708	.01901	−.00070	−.0742	.01996	−.00166
24	−.0065	.00662	.00015	.0011	.01155[a]	.00032
25	−.0012	.00418	.00005	−.0008	.01048[a]	−.00008
26	−.0103	.00583	.00034	−.0128	.01204[a]	.00034
27	−.0047	.00690	−.00027	−.0234	.01295[a]	.00041
28	−.0003	.00405	.00067	−.0217	.01791	.00232
29	.0137	.01092	.00051	.0140	.01531	.00051
30	.0037	.00523	−.00052	−.0088	.01492	−.00522
31	.0224	.01147	.00484	.0305	.01861	.00177
32	−.0088	.01312	−.00559	−.0033	.01489	.00009
33	.0001	.00422	.00001	.0116	.01320	.00036
34	−.0176	.00820	−.00010	−.0020	.00706	.00004
35	−.0043	.01011	−.00243	−.0072	.01040[a]	−.00115
36	−.0136	.01446	−.00528	−.0332	.01030[a]	−.00075
37	.0312	.00804	.00027	−.0131	.01602	.00013
38	.0145	.01158	−.00471	.0148	.00827[a]	.00001
39	−.0110	.00290[a]	−.00008	.0256	.01262[a]	−.00019
40	−.0243	.00705	−.00025	.0130	.01273	−.00093
41	.0386	.01538	.00110	.0187	.02138	−.00292
42	.0151	.00502	−.00025	.0330	.01035[a]	−.00078
43	.0111	.00812	.00016	−.0028	.01017[a]	−.00008
44	.0191	.00537	.00001	.0155	.01426	.00104

[a] Not consistent with Model I. [b] Not consistent with Model I or Model II.

third central moments of each subject for regions S and L, as indicated in Table 5. Inspection of the means makes it apparent that all individual means are close to the reinforcement means. This fact is consistent with all models. The third moments exhibit a considerable dispersion, but are equally divided into positive and negative values.

TABLE 6

VARIANCES OF THE REINFORCEMENT (OR PSEUDOREINFORCEMENT) DISTRIBUTIONS, i.e., LOWER BOUNDS FOR THE VARIANCES OF THE ASYMPTOTIC RESPONSE DISTRIBUTIONS IN THE DIFFERENT MODELS AND REGIONS

(All entries have been divided by π^2)

Bound	Model		
	I	II	III
Lower bound for region S ($l = .10\pi$)	.0033	.0017	.0008
Lower bound for region L ($l = .20\pi$)	.0133	.0067	.0033

The most revealing characteristic, however, is the variance. By means of the variance property described in Sec. 1, we can compute a priori, for each of the different models, a lower bound for the variance provided by the reinforcement or pseudoreinforcement distribution. Lower bounds for regions S and L corresponding to Models I, II (and II′), and III are exhibited in Table 6. The table makes it clear that to go from Model I to Model III is to go from the strictest model to the weakest, since a variance consistent with Model I is consistent with Models II and III, a variance not consistent with Model II is not consistent with Model I, and so on. In Table 5, the response variances not consistent with Model I are labeled *a*, and the results not consistent with either Model I or Model II are labeled *b*. In terms of percentages, in the case of region S, 92 per cent of the individual variances are consistent with Model I and 100 per cent with Models II and III. In the case of region L only 52 per cent of the individual variances are consistent with Model I, 93 per cent with Model II and 100 per cent with Model III. From these results it can be anticipated that the over-all fit of Model I will be far from satisfactory for region L.

Asymptotic response distribution. The histograms for the last 300 trials in class intervals of $.02\pi$ for regions S and L as well as those for regions S′ and L′ are shown in Table 1. Table 7 gives the empirical means and variances for the regions S, L, S′, and L′, and also the third and fourth moments for regions S and L.

Using the empirical variances shown in Table 7 and the additive variance property mentioned in the discussion of theoretical results above, we may

TABLE 7

CHARACTERISTICS OF THE EMPIRICAL RESPONSE DISTRIBUTION FOR LAST 300 TRIALS
(Means have been divided by π, variances by π^2, third moments by π^3, and fourth moments by π^4)

Property	Region			
	S	S′	L	L′
Mean	−.0015	+.0008	+.0045	−.0020
Variance	.008546	.008547	.013957	.013974
Third moment	+.00032956	—[a]	+.00010241	—[a]
Fourth moment	.00162628	—[a]	.00152945	—[a]
Number of observations	6565		6635	

[a] Not obtained.

TABLE 8

VARIANCES OF THE SMEARING DISTRIBUTIONS
(All entries have been divided by π^2)

Model	Region			
	S	S′	L	L′
I	.005212	.005214	.000624	.000640
II, II′	.006879	.006881	.007291	.007307
III	.007712	.007714	.010624	.011064

derive at once the variance of the smearing distribution for each of the models and for each of the four regions S, S′, L, and L′. The results are shown in Table 8.

The next step is to make specific assumptions about the smearing distributions in order to derive the theoretical response distributions. Following the approach made in the Suppes and Frankmann (1961) study, we have investigated the uniform and symmetric beta distributions as possible smearing distributions.

The uniform smearing distribution is characterized by one parameter, its range $2a$. The variance of the uniform distribution with parameter a is $a^2/3$. The symmetric beta smearing distribution is characterized by two parameters, the range $2b$ and the exponent n. The form of the density is as follows:

$$
\begin{aligned}
k(x, y) &= \frac{1}{bB(n+1, \frac{1}{2})}\left[1 - \frac{(x-y)^2}{b^2}\right]^n \qquad \text{for } |x - y| < b \\
&= 0 \qquad \text{for } |x - y| \geq b,
\end{aligned}
$$

where $B(n+1, \frac{1}{2})$ is the usual beta coefficient. The variance of the beta distribution with parameters b and n is $b^2/(2n+3)$.

The estimation of the single parameter for the uniform distribution presents no problem, for it may be estimated directly from the difference between the empirical variance and the variance of the reinforcement distribution. The results for the three types of models and the four types of regions are listed in Table 9.

TABLE 9

ESTIMATED HALF-RANGE a OF THE UNIFORM SMEARING DISTRIBUTION AND EXPONENT n OF THE BETA SMEARING DISTRIBUTION

(Entries have been divided by π)

Model	Region							
	S		S′		L		L′	
	a	n	a	n	a	n	a	n
I	.1251	22	.1251	22	.0433	49	.0439	49
II, II′	.1437	15	.1437	15	.1479	15	.1481	15
III	.1521	15	.1521	15	.1785	10	.1787	10

For the beta distribution the problem of estimating the two parameters is more complicated. A direct application of the method of moments would entail using the empirical fourth moment but, as is well known, the empirical fourth moment is subject to large fluctuations. If we use the expression for the variance of the beta smearing distribution, i.e., $b^2/(2n+3)$, as a constraint between b and n, we may study the different theoretical response densities corresponding to the different combinations of b and n that yield the given numerical variance for the distribution. It happens that these densities are very close to each other; in fact, they are practically indistinguishable as soon as $n \geq 3$. We have therefore chosen $\pi/2$ as a fixed value for b, which provides a sufficiently wide range for any reasonable smearing effect. We have then determined n from the variance. The a priori selection of b has the further advantage of putting the uniform and beta smearing distributions on the same basis, i.e., one parameter is estimated for each. The estimates for regions S and L are shown in Table 9.

By using the estimated values of the parameters, theoretical response densities can be computed. The analytical expressions for these densities are rather involved; they are shown in the Appendix in Tables C and D. The corresponding curves are shown in Figs. 4 and 5. The qualitative facts to be inferred from these figures are fairly direct. In the case of region S, when a uniform smearing distribution is used, we see from Fig. 4 that Models I, II, and III can be clearly discriminated from each other even at the gross level of the asymptotic response distribution. This is not true when beta smearing

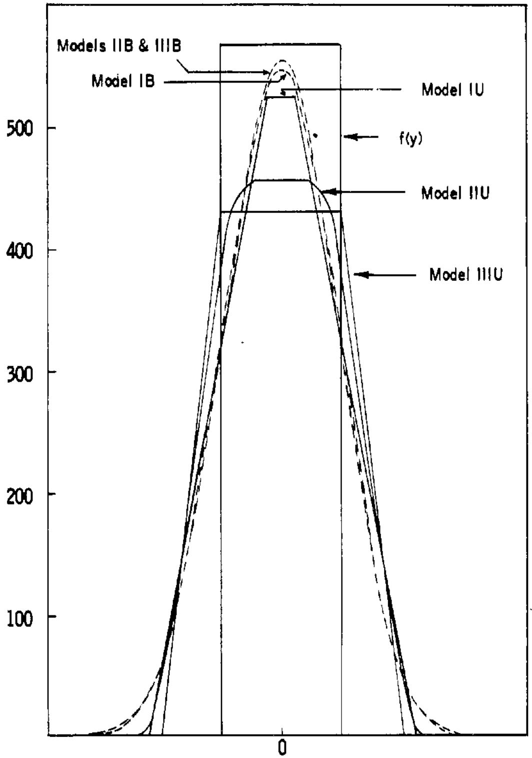

FIG. 4. Predicted asymptotic response densities derived from the various models and smearing distributions for region S.

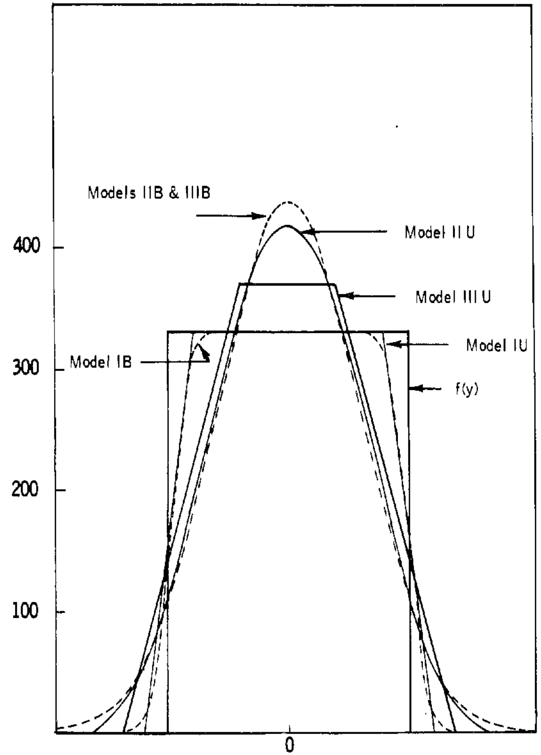

FIG. 5. Predicted asymptotic response densities derived from the various models and smearing distributions for region L.

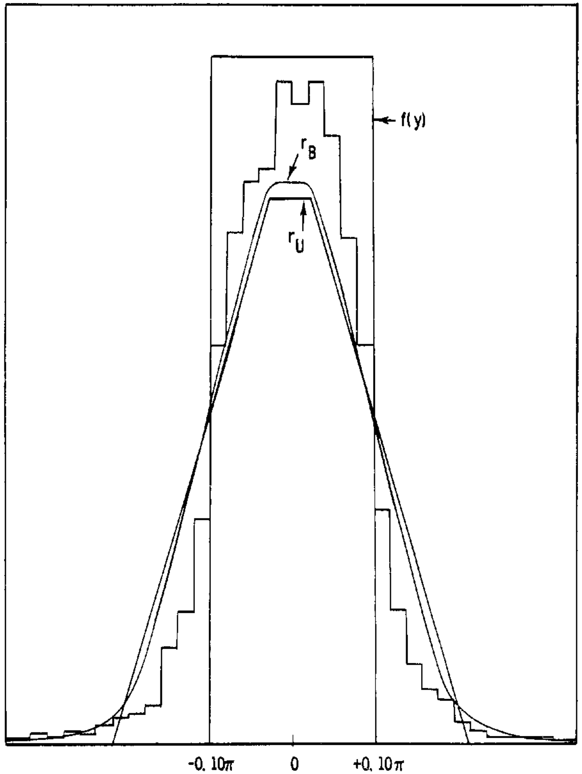

FIG. 6. Response histogram for last 300 trials for region S and two predicted densities derived from Model I.

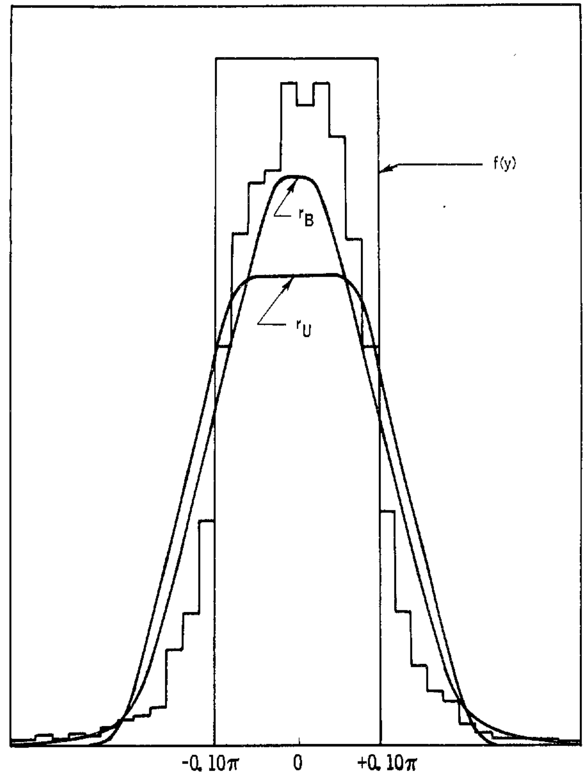

FIG. 7. Response histogram for last 300 trials for region S and two predicted densities derived from Model II.

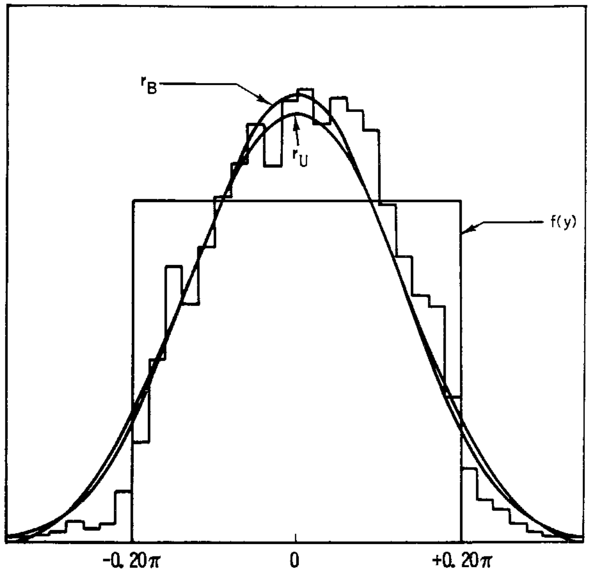

FIG. 8. Response histogram for last 300 trials for region L and two predicted densities derived from Model II.

distributions are used, as may also be seen from Fig. 4. Only one response curve with a beta smearing distribution is shown for Models II and III because of the near identity of the response distributions for these models. On the other hand, when the parameters are estimated for region L, a clear discrimination between Model I and either Model II or Model III is obtained, as may be seen from Fig. 5, even though beta smearing distributions are used. However, the curves for Models II and III are too nearly identical to be drawn separately in Fig. 5. The theoretical response distributions for region L based on a uniform smearing distribution are shown in Fig. 5, and as in the case of those shown in Fig. 4 for region S, a clear discrimination between models is possible.

Figures 6, 7, and 8 provide a direct comparison of the empirical histograms and various fitted densities. Inspection of Fig. 6 indicates that Model 7 fits the empirical histogram of region S fairly well regardless of whether I uniform or a beta smearing distribution is used. It is apparent from Fig. a that the beta smearing distribution yields a better fit to the empirical histogram than does the uniform when Model II is applied to region S. This is also true, but to a lesser extent, when Model II is applied to region L, as may be seen in Fig. 8.

When we consider goodness-of-fit tests for the asymptotic response distributions, the question of which observations should be taken arises. It is precisely a feature of the models we consider to predict dependencies between successive responses. Dividing each of the four regions into four classes and computing the χ^2 for the test of independence between successive responses for each region, we obtain for region S a χ^2 of 89.1 with 9 df, and for region L a χ^2 of 84.3 with 9 df. Both results are highly significant and indicate strong dependency between successive responses. When we take every other response and compute the corresponding χ^2's we obtain a χ^2 of 74.4 for region S and a χ^2 of 52.2 for region L. As is evident, there has been a considerable reduction in dependency of responses when only every other response is used, but these χ^2's are still significant. However, we shall perform goodness-of-fit tests using every other observation; some inflation of the χ^2's obtained may result from the remaining dependencies in the observation. The results are shown in Table 10. The first thing we note is that the fit of Models I, II, and III for the small region is quite different from the fit of these models for the large region. In the case of region S, Model I fits better than Model II or III regardless of the smearing distribution used, although the difference is small (218.2 vs. 223.0) when a beta distribution is used. On the other hand, Model I was not used for region L, because it could account for such a small percentage of individual subjects' variances, as mentioned earlier.

We conjecture that this difference between models for the two regions may be explained along the following lines. Because of the small physical size of region S, the sequence of exact positions of the reinforcements has a negligible effect when the subjects have formed the association between the general reinforcement region and the discriminating stimuli. For region L the converse holds. We find further support of this conjecture in the sequential statistics presented below.

In the Suppes and Frankmann (1961) study, it was impossible to distinguish between the assumptions of a uniform or a beta smearing distribution. As earlier remarks have indicated, this is not the case for the present study (see, for example, the remarks about Figs. 4 and 5). On the other hand, the difference is not large, as may be seen from Table 10. For Model I and

TABLE 10

CHI-SQUARE GOODNESS OF FIT OF PREDICTED ASYMPTOTIC RESPONSE DISTRIBUTIONS. DATA BASED ON EVERY OTHER RESPONSE FOR LAST 300 TRIALS

(U = uniform smearing distribution, B = beta smearing distribution, degrees of freedom of each test shown in parentheses)

Model	Region			
	S	S′	L	L′
IU	189.5 (16)	196.1 (16)	—[a]	—[a]
IIU	431.2 (16)	435.4 (16)	118.2 (18)	103.8 (18)
IB	218.2 (18)	208.1 (16)	—[a]	—[a]
II & IIIB	223.0 (18)	212.0 (16)	125.4 (18)	116.3 (18)

[a] Not obtained.

region S, as well as for Model II and region L, there is an insignificant difference between the fits of the two kinds of smearing distributions. The only significant difference occurs in the application of Model II to region S, for which the uniform smearing distribution leads to a response distribution that fits badly. The bad fit that arises from the use of the uniform smearing distribution seems to be explained by the following considerations. In Model II the pseudoreinforcement distribution is a sharp triangular distribution. For region S many of the responses at asymptote fall outside the range of this distribution. The flat uniform smearing distribution is not as able as the beta distribution to account simultaneously for the sharp peak in the middle of the response histogram and the relatively large tails.

It is clear from Table 10 that the same sort of results are obtained for regions S′ and L′ as for S and L, respectively.

Sequential statistics. The sequential statistics have been tabulated on the basis of classifying the responses for a given region into one of four intervals. This small number of classes has been chosen in order to obtain sufficiently large conditional frequencies in each cell. The cutoff points were obtained from equal divisions of the reinforcement distribution. On that basis, two kinds of sequential statistics were obtained for regions S, S′, L, and L′: the probability of a response given the last reinforcement $P(X_n \mid Y_{n-1})$, and the probability of a response given the last two reinforcements

$$P(X_n \mid Y_{n-1}, Y_{n-2}).$$

The corresponding theoretical statistics have been investigated only in the case of a hypothetical *uniform* smearing distribution, because of the considerable difficulty involved in computing these statistics when a beta distribution is used.

Calculation of the theoretical statistics requires the knowledge of the learning parameter θ. The values of θ that we used were not those obtained from the learning curves, but were those values yielding for each model the minimum χ^2 for the statistic $P(X_n \mid Y_{n-1})$. Although the values obtained from the learning curves are around .016, the values estimated from the sequential statistics range from .12 to .66. Such a discrepancy cannot be attributed to random fluctuations. It has been found in other learning experiments. How χ^2 varies as a function of θ can be inferred from Fig. 9, which gives $\chi^2 = f(\theta)$ for the fit of Model II to the data for region L.

In Table 11 are shown the observed and predicted probabilities of a response given the last reinforcement for regions S and L. In most cases, the predicted probabilities are of the right order of magnitude, but in terms of the large number of observations on which the observed frequencies are based, the discrepancies are large enough to yield sizable χ^2's in the goodness-of-fit test. The test for each model has ten degrees of freedom. The results for region S are as follows: $\chi^2(\mathrm{I}) = 185.9$, $\chi^2(\mathrm{II}) = 418.3$, and $\chi^2(\mathrm{III}) = 575.2$, with $\theta^*_{\mathrm{I}} = .260$, $\theta^*_{\mathrm{II}} = .600$, and $\theta^*_{\mathrm{III}} = .660$. Evidently Model I fits the

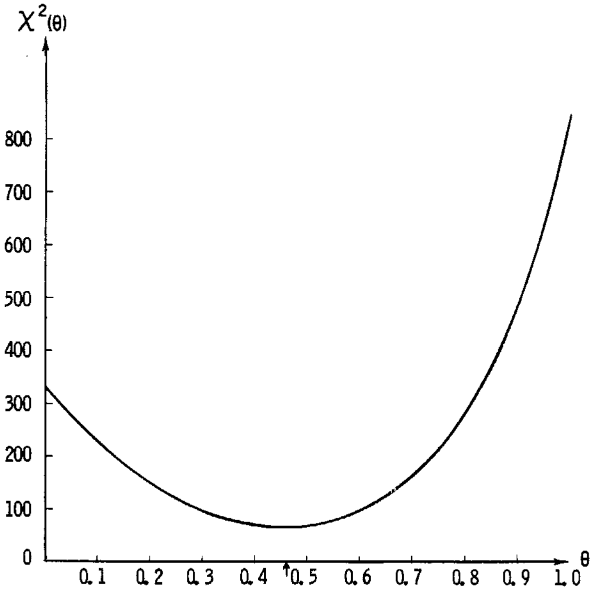

FIG. 9. Chi-square as a function of θ for the fit to region L of $P(X_n \mid Y_{n-1})$ derived from Model II.

observed frequencies by far the best of the three models we have considered here. Essentially the same results are obtained for region S′: $\chi^2(\mathrm{I}) = 187.2$, $\chi^2(\mathrm{II}) = 419.3$, and $\chi^2(\mathrm{III}) = 576.0$. One way to express the difference between Models I, II, and III is to say that as we progress from Model I to Model III, increasing account is taken of past reinforcements. The question of what results we obtain for the small regions if we completely ignore information about the preceding reinforcement naturally arises. If we consider as a fourth model (Model IV in Table 11) the one for which the probabilities are independent of the reinforcements—simply determined from the asymptotic response distribution—we somewhat surprisingly obtain a smaller χ^2 (with one more degree of freedom) than for Model I, namely, $\chi^2(\mathrm{IV}) = 147.3$ for region S, and $\chi^2(\mathrm{IV}) = 152.0$ for region S′, which confirms our earlier conjecture about the negligible effect of the exact position of reinforcements for region S.

The situation is considerably different when we turn to region L. Intuitive psychological reasons for expecting a difference are obvious, namely, the much larger reinforcement region can easily lead to subjects' discriminating between different reinforcement positions within that region. The goodness-of-fit tests for the observed and predicted probabilities shown in Table 11 are as follows: for region L, $\chi^2(\mathrm{I}) = 383.9$, $\chi^2(\mathrm{II}) = 67.8$, and $\chi^2(\mathrm{III}) = 179.4$. For region L′ the results are quite similar: $\chi^2(\mathrm{I}) = 367.3$, $\chi^2(\mathrm{II}) = 46.7$, and $\chi^2(\mathrm{III}) = 161.4$. From these results it is at once evident that Model II yields

TABLE 11

PREDICTION OF RESPONSE QUARTILES FOR REGIONS S AND L, GIVEN QUARTILE OF PRECEDING REINFORCEMENT

(The first number in each group is the observed proportion based on last 300 trials, the second the prediction of Model I, the third that of Models II and II′, the fourth that of Model III, and the fifth that of Model IV)

Y_{n-1} \ X_n	Region S				Region L			
	1	2	3	4	1	2	3	4
1	.335	.232	.221	.212	.274	.311	.252	.163
	.383	.195	.189	.233	.327	.233	.220	.220
	.404	.174	.174	.248	.305	.308	.251	.136
	.417	.164	.164	.255	.306	.278	.273	.143
	.269	.224	.240	.267	.182	.288	.317	.213
2	.299	.248	.226	.227	.190	.332	.320	.158
	.331	.195	.195	.279	.233	.314	.233	.220
	.352	.174	.174	.300	.231	.308	.287	.174
	.363	.164	.164	.309	.250	.278	.278	.194
	.269	.224	.240	.267	.182	.288	.317	.213
3	.229	.221	.266	.284	.152	.283	.358	.207
	.279	.195	.195	.331	.220	.233	.314	.233
	.300	.174	.174	.352	.174	.287	.308	.231
	.309	.164	.164	.363	.194	.278	.278	.250
	.269	.224	.240	.267	.182	.288	.317	.213
4	.215	.193	.245	.347	.127	.227	.328	.318
	.233	.189	.195	.383	.220	.220	.233	.327
	.248	.174	.174	.404	.136	.251	.308	.305
	.255	.164	.164	.417	.143	.273	.278	.306
	.269	.224	.240	.267	.182	.288	.317	.213

the best predictions for region L and is in fact considerably better than any of the predictions for region S, a result that corresponds to those given above for the asymptotic response distributions. Also, as would be expected from the fact that Model I is not as good as Model II, the model that ignores the effects of reinforcement entirely (Model IV) yields $\chi^2(\text{IV}) = 300.0$ for region L, and $\chi^2(\text{IV}) = 305.8$ for region L′. It may be noted that the qualitative ordering of the conditional probabilities in all four rows for region L in Table 11 is the same for Model II and the observed data. A similar relation does not hold for any of the models as applied to region S.

Tables 12 and 13 present the observed and predicted sequential statistics $P(X_n \mid Y_{n-1}, Y_{n-2})$. Again we observe that the predicted probabilities are usually of the right order of magnitude but that significant discrepancies do exist. The goodness-of-fit tests (with 46 df) support these observations. In the

TABLE 12

PREDICTION OF RESPONSE QUARTILE FOR REGION S, GIVEN QUARTILES OF TWO PRECEDING REINFORCEMENTS

(The first number is the observed proportion, the second the prediction of Model I, and the third that of Model III)

$Y_{n-2}\ Y_{n-1}$ \ X_n	1	2	3	4	$Y_{n-2}\ Y_{n-1}$ \ X_n	1	2	3	4
11	.321	.206	.232	.241	31	.351	.244	.210	.195
	.439	.197	.185	.179		.335	.196	.192	.277
	.445	.164	.164	.227		.336	.164	.164	.336
12	.273	.235	.255	.237	32	.313	.244	.201	.242
	.401	.196	.190	.213		.297	.197	.196	.310
	.426	.164	.164	.246		.318	.164	.164	.354
13	.229	.239	.257	.275	33	.236	.213	.284	.267
	.362	.197	.190	.251		.258	.196	.197	.349
	.408	.164	.164	.264		.300	.164	.164	.372
14	.234	.198	.237	.329	34	.187	.195	.245	.373
	.328	.192	.190	.290		.225	.192	.196	.387
	.391	.164	.164	.281		.281	.164	.164	.391
21	.367	.218	.238	.177	41	.301	.256	.209	.234
	.387	.196	.192	.224		.290	.190	.192	.328
	.391	.164	.164	.281		.281	.164	.164	.391
22	.282	.263	.215	.240	42	.330	.249	.238	.183
	.349	.197	.196	.258		.251	.190	.197	.362
	.372	.164	.164	.300		.264	.164	.164	.408
23	.227	.232	.248	.293	43	.217	.195	.279	.309
	.310	.196	.197	.297		.213	.190	.196	.401
	.354	.164	.164	.318		.246	.164	.164	.426
24	.227	.235	.226	.312	44	.224	.141	.272	.363
	.277	.192	.196	.335		.179	.185	.197	.439
	.336	.164	.164	.336		.227	.164	.164	.445

case of region S, $\chi^2(\text{I}) = 336.7$ and $\chi^2(\text{III}) = 743.1$. Because of the earlier results obtained for the statistic $P(X_n \mid Y_{n-1})$, we did not apply Model II to region S, but restricted ourselves to Models I and III. The comparisons of Models I and III for the two sequential statistics are consistent with each other. The same results obtain for region S′, for which $\chi^2(\text{I}) = 326.8$, and $\chi^2(\text{III}) = 730.1$.

All three models were applied to the statistic $P(X_n \mid Y_{n-1}, Y_{n-2})$ for regions L and L′. The goodness-of-fit results are as follows: for region L, $\chi^2(\text{I}) = 550.5$, $\chi^2(\text{II}) = 788.3$, $\chi^2(\text{II}') = 366.3$, and $\chi^2(\text{III}) = 387.8$. The most interesting thing about these results is that whereas in the case of the

TABLE 13

PREDICTIONS OF RESPONSE QUARTILE FOR REGION L GIVEN QUARTILES OF TWO PRECEDING REINFORCEMENTS

(The first number is the observed proportion, the second the prediction of Model II, the third the prediction of Model II′, and the fourth the prediction of Model III)

X_n / $Y_{n-2}\ Y_{n-1}$	1	2	3	4	X_n / $Y_{n-2}\ Y_{n-1}$	1	2	3	4
11	.251	.323	.252	.174	31	.278	.318	.254	.150
	.473	.307	.147	.073		.246	.311	.307	.136
	.422	.297	.167	.114		.266	.311	.297	.126
	.356	.279	.270	.095		.243	.279	.276	.202
12	.206	.334	.321	.139	32	.203	.337	.286	.174
	.395	.322	.210	.073		.168	.311	.322	.199
	.344	.312	.230	.114		.189	.311	.311	.189
	.322	.279	.274	.125		.210	.279	.279	.232
13	.162	.271	.343	.224	33	.146	.270	.379	.205
	.317	.322	.277	.085		.105	.296	.322	.277
	.266	.311	.297	.126		.126	.297	.311	.266
	.288	.279	.274	.159		.176	.279	.279	.266
14	.153	.197	.323	.327	34	.109	.269	.321	.301
	.239	.322	.291	.148		.094	.229	.322	.355
	.189	.311	.311	.189		.144	.230	.312	.344
	.258	.276	.273	.193		.146	.276	.279	.229
21	.301	.301	.247	.151	41	.266	.306	.250	.178
	.355	.322	.229	.094		.148	.291	.322	.239
	.344	.312	.230	.114		.189	.311	.311	.189
	.229	.279	.276	.146		.193	.273	.276	.258
22	.172	.322	.359	.147	42	.180	.338	.312	.170
	.277	.322	.296	.105		.085	.277	.322	.317
	.266	.311	.311	.189		.126	.297	.311	.266
	.266	.279	.279	.176		.159	.274	.279	.288
23	.169	.284	.336	.211	43	.129	.303	.371	.197
	.199	.322	.311	.168		.073	.210	.322	.395
	.189	.311	.311	.189		.114	.230	.312	.344
	.232	.279	.279	.210		.125	.274	.279	.322
24	.127	.228	.318	.327	44	.123	.204	.351	.322
	.136	.307	.311	.246		.073	.147	.307	.473
	.126	.297	.311	.266		.144	.167	.297	.422
	.202	.276	.279	.243		.095	.270	.279	.356

previous statistic Models II and II′ make precisely the same predictions, in the present case Model II′ is very much superior to Model II. An examination of the two models, as shown by Eqs. (13) and (14), shows that Model II assigns less weight to the mean response distribution on trial n and includes an additional term involving the reinforcement on trial $n - 1$. The difference between the predictions of Models II and II′ is given by the expression

$$\theta(1 - \theta)\left[\frac{H'(X_n, Y_{n-1})}{F(Y_n)} - R_{\mathrm{II}}(X_n)\right].$$

We may also note that the goodness-of-fit results for region L′ reflect precisely the same qualitative conclusions: $\chi^2(\mathrm{I}) = 542.9$, $\chi^2(\mathrm{II}) = 777.1$, $\chi^2(\mathrm{II}') = 352.5$, and $\chi^2(\mathrm{III}) = 377.7$.

It may also be noted that in the case of both L and L′, the relative fits of Models II′ and III are essentially the same. These results suggest that even for the large regions subjects' responses are not strongly affected by the second preceding reinforcement, for it will be remembered that Model III replaces the second preceding reinforcement Y_{n-2} by the mean of the reinforcement distribution.

Several general remarks can be made about these sequential statistics. In the first place, they constitute a crucial test of the theory. As the reinforcement dependencies are increased, these statistics approach a sufficient statistic for the stochastic process defined by the theory. Moreover, it is the consideration of the sequential statistics that sharply distinguishes between different kinds of models. It is possible to describe a wide variety of models that will lead to the same asymptotic response distributions but that differ significantly in sequential predictions. The results obtained are not as good as we had hoped for, but they do differ sufficiently for the different models in order clearly to discriminate between them. As was stated in the beginning of this paper, the obtained values of χ^2 are magnified both by the very large number of observations on which they are based and by the fact that the sequential dependencies used here are not of sufficient length to make the responses themselves statistically independent. On the other hand, these two observations are not sufficient to explain the large values reported. There is clearly a significant gap between theory and observation, and it appears that it will be necessary to revise the theory proposed in this paper in order to close this gap adequately.

(*References follow Appendix, p.* 357.)

Appendix

The main purpose of this Appendix is to derive the results presented in the section on the theoretical results. The material in the Appendix is also organized to make explicit the mathematical details involved in making an application of the general formulas given in the body of the paper to experimental data. These details are elementary but quite tedious to work out. We hope that their inclusion will encourage other investigators to conduct additional experiments.

We consider the two following probability densities, defined on the real line:

i. A density $f(y)$, which we interpret as a noncontingent reinforcement or pseudoreinforcement density, as the case may be.

ii. A smearing density $k(x; y)$. The number y is a location parameter, that we can take to be for example, the mean. We further assume $k(x; y)$ to be translation invariant, i.e., we suppose that

$$k(x; y) = k(x + y' - y; y')$$

for all y, y', which implies that $k(x; y) = k(x - y; 0)$.

As a result, all derivations can be made using only the density $k(t; 0)$, which we write $k(t)$ for simplicity. In fact we use in the sequel the same symbol k for both the function of two variables previously considered and the functions of one variable just defined; they are related by

$$k(x; y) = k(x - y). \tag{A-1}$$

All results in this Appendix are given in terms of f and k.

The results are organized into five categories: additive variance property and moment relations; asymptotic response distribution when f and k are constrained; reinforcement-dependent statistics; linear models and stimulus-sampling models; and calculation of sequential statistics.

A-1. Additive variance property and moment relations

We are first concerned with the asymptotic response density $r(x)$, given for all models [$f(y)$ being properly interpreted] by the formula

$$r(x) = \int k(x - y) f(y)\, dy. \tag{A-2}$$

(The limits of integration here and in what follows are always from $-\infty$ to ∞ unless explicitly counterindicated.) In probability terms, Eq. (A-2) expresses that the density r is the convolution of the two densities k and f. It is classical (Robbins, 1948) that this fact is equivalent to a relation between characteristic functions, or else to a sequence of relations between the moments of the distributions (see, e.g., Cramér, 1946, p. 191). If μ_i is the central moment of order i, we have

$$\mu_1(r) = \mu_1(k) + \mu_1(f), \tag{A-3}$$

$$\mu_2(r) = \mu_2(k) + \mu_2(f), \tag{A-4}$$

$$\mu_3(r) = \mu_3(k) + \mu_3(f), \tag{A-5}$$

$$\mu_4(r) = \mu_4(k) + \mu_4(f) + 6\, \mu_4(k)\mu_4(f), \tag{A-6}$$

and so on. Eq. (A-5) implies that the asymptotic response distribution is symmetrical if the smearing distribution and the reinforcement distribution are. Equation (A-6) could be applied practically if the sample fourth moment offered more statistical stability. Equation (A-4) is the additive variance property. Of course the variance property can be proved in a direct and more elementary way. Assuming that the *mean* of the smearing distribution $k(x; y)$ is $x = y$, i.e.,

$$\int xk(x; y)\, dx = y,$$

we first prove that the mean of the distribution $r(x)$ coincides with the mean of the reinforcement distribution:

$$\int xr(x)dx = \int\int xk(x; y)f(y)\, dydx$$

$$= \int f(y)\left[\int xk(x; y)\, dx\right] dy = \int f(y)\, ydy.$$

This mean can be taken to be 0 without loss of generality. We then have

$$\operatorname{var}(f) = \int y^2 f(y)\, dy \quad \text{and} \quad \operatorname{var}(k) = \int x^2 k(x; y)\, dx - y^2,$$

or

$$\int x^2 k(x; y)\, dx = \operatorname{var}(k) + y^2.$$

Now

$$\operatorname{var}(r) = \int x^2 r(x)\, dx = \int\int x^2 k(x; y) f(y)\, dxdy$$

$$= \int f(y)\left[\int x^2 k(x; y)\, dx\right] dy$$

$$= \int f(y) \operatorname{var}(k)\, dy + \int f(y) y^2\, dy$$

$$= \operatorname{var}(k) + \operatorname{var}(f).$$

A-2. Asymptotic response density with constraints on the distributions

In this paragraph we specialize Eq. (A-2) by introducing additional constraints on f and k. We first assume that (i), $f(y)$ is symmetric around 0, and that (ii), $k(x; y)$ is symmetric around $x = y$.

One can easily check that (ii) implies: $k(x; y) = k(y; x) = k(-x; -y) = k(-y; -x)$. A consequence of (i) and (ii) is that the density $r(x)$ is also symmetric around 0. We further assume that (iii), $f(y) = 0$ outside the interval $(-l, +l)$, and that (iv), $k(t) = 0$ outside the interval $(-a, +a)$. It follows from (iii) and (iv) that $r(x) = 0$ outside $(-a - l, a + l)$. The assumption of (i)–(iv) yields different formulas for $r(x)$ according to the different

TABLE A-1
Asymptotic Response Distribution with f and k Satisfying Conditions (i)–(iv)

I. General formula: $r(x) = \int_{-\infty}^{+\infty} k(t)f(x-t)\,dt$

II. Assume that $f(y)$ has range $2l$ and that $k(t)$ has range $2a$. The formulas for $r(x)$ are as follows:

$$
\begin{array}{lll}
r(x) = 0 & \text{for } x \geq a + l & \\
r(x) = \int_{x-l}^{x+l} k(t)f(x-t)\,dt & \text{for } 0 \leq x \leq a-l, & a \geq l; \\
r(x) = \int_{x-l}^{a} k(t)f(x-t)\,dt & \text{for } a-l \leq x \leq a+l, & a \geq l; \\
r(x) = \int_{-a}^{+a} k(t)f(x-t)\,dt & \text{for } 0 \leq x \leq l-a, & a \leq l; \\
r(x) = \int_{x-l}^{+a} k(t)f(x-t)\,dt & \text{for } l-a \leq x \leq l+a, & a \leq l.
\end{array}
$$

III. Assume further:

$$
f(y) = \begin{cases} f_+(y) & \text{when } y \geq 0, \\ f_-(y) & \text{when } y < 0 \end{cases} \quad \text{and} \quad k(t) = \begin{cases} k_+(t) & \text{when } t \geq 0, \\ k_-(t) & \text{when } t < 0. \end{cases}
$$

In the formulas k_+, k_-, f_+, f_- stand for $k_+(t), k_-(t), f_+(x-t), f_-(x-t)$. The integrals are taken over t.

Case 1 ($a \geq 2l$)	[*Cases 1* and *2* coincide when $k_+ = k_-$.]	*Case 2* ($l \leq a \leq 2l$)
$0 \leq x \leq l$	$\int_{x-l}^{0} k_-f_+ + \int_0^x k_+f_+ + \int_x^{x+l} k_+f_-$	$0 \leq x \leq a-l$
$l \leq x \leq a-l$	$\int_{x-l}^{x} k_+f_+ + \int_x^{x+l} k_+f_-$	
	$\int_{x-l}^{0} k_-f_+ + \int_0^x k_+f_+ + \int_x^{a} k_+f_-$	$a-l \leq x \leq l$
$a-l \leq x \leq a$	$\int_{x-l}^{x} k_+f_+ + \int_x^{a} k_+f_-$	$l \leq x \leq a$
$a \leq x \leq a+l$	$\int_{x-l}^{a} k_+f_+$	$a \leq x \leq a+l$

Case 3 ($a \leq l \leq 2a$)	[*Cases 3* and *4* coincide when $f_+ = f_-$.]	*Case 4* ($l \geq 2a$)
$0 \leq x \leq l-a$	$\int_{-a}^{0} k_-f_+ + \int_0^x k_+f_+ + \int_x^{a} k_+f_-$	$0 \leq x \leq a$
	$\int_{-a}^{0} k_-f_+ + \int_0^a k_+f_+$	$a \leq x \leq l-a$
$l-a \leq x \leq a$	$\int_{x-l}^{0} k_-f_+ + \int_0^x k_+f_+ + \int_x^{a} k_+f_-$	
$a \leq x \leq l$	$\int_{x-l}^{0} k_-f_+ + \int_0^a k_+f_+$	$l-a \leq x \leq l$
$l \leq x \leq a+l$	$\int_{x-l}^{a} k_+f_+$	$l \leq x \leq a+l$

values of a and l. Table A-1 presents the different cases. Table A-1 also presents the formulas to apply when $f(y)$ and $k(t)$ have different analytical formulations according to the sign of the argument. Thus f_+ means that $y \geq 0$ in $f(y)$, f_- means that $y \leq 0$, k_+ means that $t \geq 0$ in $k(t)$ and k_- that $t \leq 0$. Because the analytical expressions coincide for various cases under Part III of the table, conditions on a, l, and x are shown to both the left and the right. Thus the first expression in Part III holds for case 1 when $0 \leq x \leq l$ and for case 2 when $0 \leq x \leq a - l$. In the second line no condition on x is shown under case 2, and then this expression does not apply to case 2. It is understood that each expression gives $r(x)$ for the restriction indicated.

Tables A-2, A-3, and A-4 are specifications of Table A-1 using the reinforcement (or pseudoreinforcement) density $f(y)$ corresponding to Models I, II, III, and a uniform or beta smearing distribution. Table A-2 gives the asymptotic response distribution for Models I and II with the smearing distribution arbitrary except for the satisfaction of conditions (i)–(iv) and the assumption that $a > 2l$. The density $r(x)$ has three distinct expressions for Model I, and four for Model II, depending on the relation holding between a, l, and x. For notational convenience the expression for $2lr(x)$ rather than $r(x)$ is shown for Model I and $l^2r(x)$ is shown for Model II.

TABLE A-2

ASYMPTOTIC RESPONSE DISTRIBUTION FOR ARBITRARY SMEARING DISTRIBUTION AND UNIFORM REINFORCEMENT DISTRIBUTION

MODEL I: $f(y)$ is uniform density over $(-l + l)$, $a > 2l$;

$$2lr(x) = \int_0^{l-x} k(t)\,dt + \int_0^{x+l} k(t)\,dt \quad \text{for } 0 \leq x \leq l;$$

$$2lr(x) = \int_{x-l}^{x+l} k(t)\,dt \quad \text{for } l \leq x \leq a - l;$$

$$2lr(x) = \int_{x-l}^{a} k(t)\,dt \quad \text{for } a - l \leq x \leq a + l.$$

MODEL II: Pseudoreinforcement is triangular density over $(-l + l)$;

$$l^2r(x) = \int_0^{l-x} k(t)(l - x - t)\,dt + \int_0^{x} k(t)(l - x + t)\,dt + \int_x^{x+l} k(t)(l + x - t)\,dt \quad \text{for } 0 \leq x \leq l,$$

$$l^2r(x) = \int_{x-l}^{x} k(t)(l - x + t)\,dt + \int_x^{x+l} k(t)(l + x - t)\,dt \quad \text{for } l \leq x \leq a - l,$$

$$l^2r(x) = \int_{x-l}^{x} k(t)(l - x + t)\,dt + \int_x^{a} k(t)(l + x - t)\,dt \quad \text{for } a - 1 \leq x \leq a,$$

$$l^2r(x) = \int_{x-l}^{a} k(t)(l - x + t)\,dt \quad \text{for } a \leq x \leq l + a.$$

TABLE A-3

ASYMPTOTIC RESPONSE DISTRIBUTION FOR UNIFORM SMEARING DISTRIBUTION ON INTERVAL $(-a,a)$ AND UNIFORM REINFORCEMENT DISTRIBUTION ON INTERVAL $(-l,l)$

MODEL I:		
Case 1. $a \geq l$	$r(x) = \frac{1}{2a}$	for $0 \leq x \leq a - l$,
	$r(x) = \frac{a + l - x}{4al}$	for $a - l \leq x < a + l$,
	$r(x) = 0$	for $x \geq a + l$.
Case 2. $a < l$	$r(x) = \frac{1}{2l}$	for $0 \leq x \leq l - a$,
	$r(x) = \frac{a + l - x}{4al}$	for $l - a \leq x \leq l + a$,
	$r(x) = 0$	for $x \geq l + a$.
MODEL II:		
Case 1. $a \geq l$	$r(x) = \frac{1}{2a}$	for $0 \leq x \leq a - l$,
	$r(x) = \frac{1}{2a} - \frac{[x - (a - l)]^2}{4al^2}$	for $a - l \leq x \leq a$,
	$r(x) = \frac{1}{4al^2}[x - (a + l)]^2$	for $a \leq x \leq a + l$,
	$r(x) = 0$	for $x \geq a + l$.
Case 2. $a < l < 2a$	$r(x) = \frac{-x^2}{2al^2} + \frac{2l - a}{2l^2}$	for $0 \leq x \leq l - a$,
	$r(x) = \frac{1}{2a} - \frac{[x + l - a]^2}{4al^2}$	for $l - a \leq x \leq a$,
	$r(x) = \frac{1}{4al^2}[x - (l + a)]^2$	for $a \leq x \leq a + l$,
	$r(x) = 0$	for $x \geq a + l$.
Case 3. $2a \leq l$	$r(x) = \frac{-x^2}{2al^2} + \frac{2l - a}{2l^2}$	for $0 \leq x \leq a$,
	$r(x) = \frac{l - x}{l^2}$	for $a \leq x \leq l - a$,
	$r(x) = \frac{1}{4al^2}[x - (a + l)]^2$	for $l - a \leq x \leq l + a$,
	$r(x) = 0$	for $x \geq l + a$.

Table A-3 gives the asymptotic response distribution when a uniform smearing distribution is assumed. Results for Model III may be obtained from those for Model I by replacing l with $l/2$.

Table A-4 corresponds to Table A-3 with a symmetric beta smearing distribution now assumed. The formulas given in the table have been simplified by using the following abbreviations for terms that arise naturally with the beta smearing distribution with exponent n:

$$k(t) = \begin{cases} \frac{1}{aB}\left(1 - \frac{t^2}{a^2}\right)^n & \text{for } |t| < a \\ 0 & \text{for } |t| \geq a \end{cases}$$

(Here we have used a rather than b for the half-range of the beta distribution.) The abbreviations are:

$$B = B(\tfrac{1}{2}, n+1) = B(n+1, \tfrac{1}{2}) = \int_0^1 v^{-1/2}(1-v)^n \, dv,$$

and

$$B' = B(1, n+1) = B(n+1, 1) = \int_0^1 v^n \, dv = \frac{1}{n+1}.$$

For $0 < u < 1$,

$$I(u) = \frac{1}{B} B_u\left(\frac{1}{2}, n+1\right) = \frac{1}{B}\int_0^u v^{-1/2}(1-v)^n \, dv,$$

$$I'(u) = \frac{1}{B'} B_u(1, n+1) = \frac{1}{B'}\int_0^u (1-v)^n \, dv = 1 - (1-u)^{n+1},$$

$$I(0) = 0 \text{ and } I(1) = 1.$$

For application of the results we also note the useful relations for $0 < \alpha < 1$:

$$\frac{B}{2} I(\alpha^2) = \int_0^\alpha (1-v^2)^n \, dv \qquad B'I'(\alpha^2) = \int_0^{\alpha^2} (1-v) \, dv.$$

TABLE A-4

ASYMPTOTIC RESPONSE DISTRIBUTION FOR SYMMETRIC BETA SMEARING DISTRIBUTION ON INTERVAL $(-a,a)$ AND UNIFORM REINFORCEMENT DISTRIBUTION ON INTERVAL $(-l,l)$

MODEL I:

$$r(x) = \frac{1}{4l}\left[I\left(\frac{l-x}{a}\right)^2 + I\left(\frac{l+x}{a}\right)^2\right] \qquad \text{for } 0 \leq x \leq l,$$

$$r(x) = \frac{1}{4l}\left[I\left(\frac{x-l}{a}\right)^2 - I\left(\frac{x-l}{a}\right)^2\right] \qquad \text{for } l \leq x \leq a-l,$$

$$r(x) = \frac{1}{4l}\left[1 - I\left(\frac{x-l}{a}\right)^2\right] \qquad \text{for } a-l \leq x \leq a+l,$$

$$r(x) = 0 \qquad \text{for } x \geq a+l.$$

MODEL II:

$$r(x) = \frac{1}{2l^2}\left[(l-x)I\left(\frac{l-x}{a}\right)^2 - 2xI\left(\frac{x}{a}\right)^2 + (l+x)I\left(\frac{l+x}{a}\right)^2 + a\frac{B'}{B}\left(-I'\left(\frac{l-x}{a}\right)^2 + 2I'\left(\frac{x}{a}\right)^2 - I'\left(\frac{l+x}{a}\right)^2\right)\right] \qquad 0 \leq x \leq l;$$

$$r(x) = \frac{1}{2l^2}\left[(x-l)I\left(\frac{x-l}{a}\right)^2 - 2xI\left(\frac{x}{a}\right)^2 + (x+l)I\left(\frac{x+l}{a}\right)^2 + a\frac{B'}{B}\left(-I'\left(\frac{x-l}{a}\right)^2 + 2I'\left(\frac{x}{a}\right)^2 - I'\left(\frac{x+l}{a}\right)^2\right)\right], \qquad l \leq x \leq a-l;$$

$$r(x) = \frac{1}{2l^2}\left[(x-l)I\left(\frac{x-l}{a}\right)^2 - 2xI\left(\frac{x}{a}\right)^2 + x + l + a\frac{B'}{B}\left(-I'\left(\frac{x-l}{a}\right)^2 + 2I'\left(\frac{x}{a}\right)^2 - 1\right)\right], \qquad a-l \leq x \leq a;$$

$$r(x) = \frac{1}{2l^2}\left[(x-l)\left[I\left(\frac{x-l}{a}\right) - 1\right] + \frac{aB'}{B}\left[1 - I'\left(\frac{x-l}{a}\right)^2\right]\right], \qquad a \leq x \leq a+l;$$

$$r(x) = 0, \qquad x \geq a+l.$$

A-3. Reinforcement-dependent statistics

We use the term "reinforcement-dependent" to refer to those sequential statistics that depend only on past reinforcements and not on preceding responses. The two examples considered in the main body of the paper are $P(X_n \mid Y_{n-1})$ and $P(X_n \mid Y_{n-1}, Y_{n-2})$. We have already observed that these two statistics are the same for all models in either the linear or the stimulus-sampling formulation, and it is possible to prove this observation as a general result for all reinforcement-dependent statistics in the noncontingent case. To illustrate the methods of proof we derive here the expression for $P(X_n \mid Y_{n-1}, Y_{n-2})$ for Model II in the two formulations.

Linear Model II. The basic axiom for this model is given as Eq. (6) in Sec. 1 of the paper. By simple iteration we obtain at once the recursion:

$$j_n(x_n \mid y_{n-1}, y_{n-2}, y_{n-3}, s_{n-4}) = (1-\theta)^2 j_{n-2}(x_n \mid y_{n-3}, s_{n-4}) + \theta(1-\theta)k\left(x_n; \frac{y_{n-2}+y_{n-3}}{2}\right) + \theta k\left(x_n; \frac{y_{n-1}+y_{n-2}}{2}\right).$$

We now use this recursion to calculate the joint density $j_n(x_n, y_{n-1}, y_{n-2})$. The method is the standard one of expanding, conditionalizing, applying the recursion, and then simplifying:

$$\begin{aligned} j_n(x_n, y_{n-1}, y_{n-2}) &= \iint j_n(x_n, y_{n-1}, y_{n-2}, y_{n-3}, s_{n-4})\, dy_{n-3} ds_{n-4} \\ &= \iint j_n(x_n \mid y_{n-1}, y_{n-2}, y_{n-3}, s_{n-4}) f(y_{n-1}) \\ &\qquad \cdot f(y_{n-2}) j_{n-3}(y_{n-3}, s_{n-4})\, dy_{n-3} ds_{n-4} \\ &= f(y_{n-1}) f(y_{n-2}) \Big[(1-\theta)^2 \iint j_{n-2}(x_n \mid y_{n-3}, s_{n-4}) \\ &\qquad \cdot j_{n-3}(y_{n-3}, s_{n-4})\, dy_{n-3} ds_{n-4} + (1-\theta) \\ &\qquad \cdot \theta \iint k\left(x_n; \frac{y_{n-2}+y_{n-3}}{2}\right) f_{n-3}(y_{n-3}) j_{n-4}(s_{n-4})\, dy_{n-3} ds_{n-4} \\ &\qquad + \theta \iint k\left(x_n; \frac{y_{n-1}+y_{n-2}}{2}\right) j_{n-3}(y_{n-3}, s_{n-4})\, dy_{n-3} ds_{n-4} \Big]. \end{aligned}$$

Integrating out, noting that the first term on the right is simply $r_{\mathrm{II},n}(x)$, and taking the limit at asymptote, we obtain

(A-7)

$$\lim_{n\to\infty} j(x_n, y_{n-1}, y_{n-2}) = f(y_{n-1}) f(y_{n-2}) \Big[(1-\theta)^2 r_{\mathrm{II}}(x_n) + (1-\theta) \cdot \theta k'(x_n; y_{n-2}) + \theta k\left(x_n; \frac{y_{n-1}+y_{n-2}}{2}\right) \Big],$$

where

$$k'(x; y) = \int k\left(x; \frac{y+u}{2}\right) f(u)\, du.$$

Integrating now over the appropriate intervals, and dividing by

$$P(Y_{n-1})P(Y_{n-2}),$$

we get Eq. (13) of Sec. 1.

Stimulus-sampling one-element Model II. The axioms for the stimulus-sampling models are given in Suppes (1960). A modification in the conditioning axiom (C2) concerning the mode of the smearing is required to take account of the two preceding reinforcements. The new axiom C2′ reads as follows: "If a stimulus is sampled on a trial, the mode of its smearing distribution becomes, with probability c, the mean of the two preceding reinforcements; with probability $1 - c$ the mode remains unchanged." The other axioms need no modification to accommodate the intuitive idea of Model II.

To represent the discontinuous conditioning process described by Axiom C2′ we use the Dirac delta function. Thus if f is a continuous function,

$$\int \delta(z - y)f(y)\,dy = f(z).$$

Use of this function permits a simple algorithmic derivation of complicated expressions. (We omit subscripts n and $n - 1$ on j.)

$$\begin{aligned}
&j(x_n, y_{n-1}, y_{n-2}) \\
&\quad = \int\int j(x_n, z_n, y_{n-1}, z_{n-1}, y_{n-2})\,dz_n dz_{n-1} \\
&\quad = \int\int j(x_n \mid z_n, y_{n-1}, z_{n-1}, y_{n-2}) j(z_n \mid y_{n-1}, z_{n-1}, y_{n-2}) j(y_{n-1}) \\
&\qquad\qquad \cdot j(z_{n-1}, y_{n-2})\,dz_n dz_{n-1} \\
&\quad = \int\int k(x_n; z_n)\left[c\delta\left(z_n - \frac{y_{n-1} + y_{n-2}}{2}\right) + (1 - c)\delta(z_n - z_{n-1})\right] j(y_{n-1}) \\
&\qquad\qquad \cdot j(z_{n-1}, y_{n-2})\,dz_n dz_{n-1}
\end{aligned}$$

by Axiom C2′. Then, simplifying,

$$\begin{aligned}
&j(x_n, y_{n-1}, y_{n-2}) \\
&\quad = c\int k\left(x_n; \frac{y_{n-1} + y_{n-2}}{2}\right) j(y_{n-1}) j(z_{n-1}, y_{n-2})\,dz_{n-1} \\
&\qquad + (1 - c)\int k(x_n; z_{n-1}) j(y_{n-1}) j(z_{n-1}, y_{n-2})\,dz_{n-1} \\
&\quad = ck\left(x_n; \frac{y_{n-1} + y_{n-2}}{2}\right) f(y_{n-1}) f(y_{n-2}) \\
&\qquad + (1 - c)\int k(x_n; z_{n-1}) j(y_{n-1}) j(z_{n-1}, y_{n-2})\,dz_{n-1}.
\end{aligned}$$

Consider now the second term:

$$\int k(x_n; z_{n-1})j(y_{n-1})j(z_{n-1}, y_{n-2})\, dz_{n-1}$$

$$= \int\int\int k(x_n; z_{n-1})j(y_{n-1})j(z_{n-1}, y_{n-2}, z_{n-2}, y_{n-3})\, dz_{n-1}dz_{n-2}dy_{n-3}$$

$$= \int\int\int k(x_n; z_{n-1})f(y_{n-1})j(z_{n-1} \mid y_{n-2}, z_{n-2}, y_{n-3})f(y_{n-2})$$
$$j(z_{n-2}, y_{n-3})\, dz_{n-1}dz_{n-2}dy_{n-3}$$

$$= \int\int\int k(x_n; z_{n-1})f(y_{n-1})\Big[c\delta\Big(z_{n-1} - \frac{y_{n-2}+y_{n-3}}{2}\Big)$$
$$+ (1-c)\delta(z_{n-1} - z_{n-2})\Big]\, f(y_{n-2})j(z_{n-2}, y_{n-3})\, dz_{n-1}dz_{n-2}dy_{n-3}$$

$$= c\int k\Big(x_n; \frac{y_{n-2}+y_{n-3}}{2}\Big)f(y_{n-1})f(y_{n-2})f(y_{n-3})\, dy_{n-3}$$
$$+ (1-c)\int k(x_n; z_{n-2})f(y_{n-1})f(y_{n-2})j(z_{n-2})\, dz_{n-2}.$$

Thus substituting and taking the limit as $n \longrightarrow \infty$

$$\lim_{n\to\infty} j(x_n, y_{n-1}, y_{n-2})$$
$$= cf(y_{n-1})f(y_{n-2})k\Big(x_n; \frac{y_{n-1}+y_{n-2}}{2}\Big)$$
$$+ c(1-c)f(y_{n-1})f(y_{n-2})k'(x_n; y_{n-2})$$
$$+ (1-c)^2 f(y_{n-1})f(y_{n-2})r_{11}(x_n),$$

and since this is simply the same as Eq. (7) for the linear model, the equivalence of the two models as far as the statistic $P(X_n \mid Y_{n-1}, Y_{n-2})$ is concerned is apparent.

Detailed calculations of the sequential statistics are considerably more tedious than the calculations described above for the asymptotic response distributions. We shall not give further details here, but we will sketch the procedure for calculating the function $H'(X, Y)$ which enters in several places [see Eq. (11)]. We first need

$$k'(x, y) = \int k\Big(x; \frac{y+u}{2}\Big)f(u)\, du.$$

Because we assume $k(t)$ to be of range $2a$ and $f(y)$ to be of range $2l$, the function $k'(x; y)$ is nonzero only when the following two inequalities obtain:

$$-l < u < +l, \qquad 2x - y - 2a < u < 2x - y + 2a.$$

These inequalities define a parallelogram in the (x, y) plane on which $k'(x, y)$ is nonvanishing. The straight lines defined by the equations

$$2x - y = -2a + l \quad \text{and} \quad 2x - y = 2a - l$$

divide the parallelogram into three regions, over which the limits of integration differ. If we now specialize to uniform distributions for k and f, we obtain $k'(x, y)$ for each of the three regions of the parallelogram. We then compute $H'(X, Y)$ as the double integral of $k'(x, y)f(y)$. Since we consider four intervals for X and four for Y, we must calculate $H'(X, Y)$ for 16 regions in the parallelogram on which $k'(x, y)$ is nonvanishing. By arguments from symmetry this number may be reduced to 8. Unfortunately, different cases arise according to the relative values of a and l.

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