Boolean games revisited: compact preference representation in games

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An Example

We consider here a Boolean n-players version of the well-known prisoners' dilemma. n prisoners (denoted by $1, \ldots, n$) are kept in separate cells. The same proposal is made to each of them:
"Either you **cover** your accomplices (C_i, $i = 1, \ldots, n$) or you **denounce** them ($\neg C_i$, $i = 1, \ldots, n$).

- Denouncing makes you freed while your partners will be sent to prison (except those who denounced you as well: these ones will be freed too).
- if none of you chooses to denounce, everyone will be freed."

Representation of this game in normal form for $n = 3$:

$3 : C_3$		
1 \ 2	C_2	$\overline{C}_2$
C_1	(1, 1, 1)	(0, 1, 0)
$\overline{C}_1$	(1, 0, 0)	(1, 1, 0)

$3 : \overline{C}_3$		
1 \ 2	C_2	$\overline{C}_2$
C_1	(0, 0, 1)	(0, 1, 1)
$\overline{C}_1$	(1, 0, 1)	(1, 1, 1)

Compact representation: $G = (A, V, \pi, \Phi)$ with

- $A = \{1, 2, \ldots, n\}$, set of players,
- $V = \{C_1, \ldots, C_n\}$, set of propositional variables,
- $\forall i \in \{1, \ldots, n\}, \pi_i = \{C_i\}$, control assignment function, and
- $\forall i \in \{1, \ldots, n\}, \varphi_i = \{(C_1 \wedge C_2 \wedge \ldots \wedge C_n) \vee \neg C_i\}$, utility functions.

Main Notions

Pure-strategy Nash equilibria (PNE)

A PNE is a strategy profile such that each player's strategy is an optimum response to the other players' strategies. $S = \{s_1, \ldots, s_n\}$ is a **pure-strategy Nash equilibrium** if and only if:

$$\forall i \in \{1, \ldots, n\}, \forall s'_i \in 2^{\pi_i}, u_i(S) \geq u_i(s_{-i}, s'_i)$$

The 3-**players version of prisoners' dilemma** has 2 PNE: $\{C_1C_2C_3\}$ and $\{\overline{C_1C_2C_3}\}$.

Characterization of PNE:
S is a PNE for G if and only if: $S \models \bigwedge_i (\varphi_i \vee (\neg \exists i : \varphi_i))$

Complexity: Deciding whether there is a PNE in a Boolean game is Σ_2^p-complete.

Dominated strategies

Let $s_i \in 2^{\pi_i}$ be a strategy for player i.

- s_i is **strictly dominated** if there exists another strategy s'_i such that, whatever the strategies of the other players, s'_i assures to player i a strictly bigger utility than s_i: $\exists s'_i \in 2^{\pi_i}$ s.t. $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$.
- s_i is **weakly dominated** if $\exists s'_i \in 2^{\pi_i}$ s.t. $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in 2^{\pi_{-i}}$ s.t. $u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$.

Elimination of dominated strategies in n**-players version of prisoners' dilemma** gives one result: $\{\overline{C_1C_2C_3}\}$.

Characterization of dominated strategies:

- s_i **strictly dominates** strategy s'_i if and only if: $s_i \models (\neg \exists -i : \neg \varphi_i)$ and $s'_i \models (\neg \exists -i : \varphi_i)$.
- s_i **weakly dominates** strategy s'_i if and only if: $(\varphi_i)_{s'_i} \models (\varphi_i)_{s_i}$ and $(\varphi_i)_{s_i} \not\models (\varphi_i)_{s'_i}$.

Complexity: Deciding whether a given strategy s'_i is weakly dominated is Σ_2^p-complete.

Introduction of preferences.

Let $Pref_G = \langle \succeq_1, \ldots, \succeq_n \rangle$ a collection of preference relations.

- S is a **weak PNE** (WPNE) for G iff $\forall i \in \{1, \ldots, n\}, \forall s'_i \in 2^{\pi_i}, (s'_i, s_{-i}) \not\succ_i (s_i, s_{-i})$
- S is a **strong PNE** (SPNE) for G iff $\forall i \in \{1, \ldots, n\}, \forall s'_i \in 2^{\pi_i}, (s'_i, s_{-i}) \preceq_i (s_i, s_{-i})$
- $NE_{strong}(G)$ and $NE_{weak}(G)$ denote respectively the set of strong and weak PNEs for G.

2 cases

Prioritized goals

A **prioritized goal base** Σ is a collection $\langle \Sigma^1; \ldots; \Sigma^p \rangle$ of sets of propositional formulas.

- Σ^j: set of goals of priority j,
- the smaller j, the more prioritary the formulas in Σ^j.

Discrimin preference relation $S \succ_i^{disc} S'$ iff $\exists k \in \{1, \ldots, p\}$ such that: $Sat(S, \Sigma^k) \supset Sat(S', \Sigma^k)$ and $\forall j < k$, $Sat(S, \Sigma^j) = Sat(S', \Sigma^j)$

Leximin preference relation $S \succ_i^{lex} S'$ iff $\exists k \in \{1, \ldots, p\}$ such that: $|Sat(S, \Sigma^k)| > |Sat(S', \Sigma^k)|$ and $\forall j < k$, $|Sat(S, \Sigma^j)| = |Sat(S', \Sigma^j)|$.

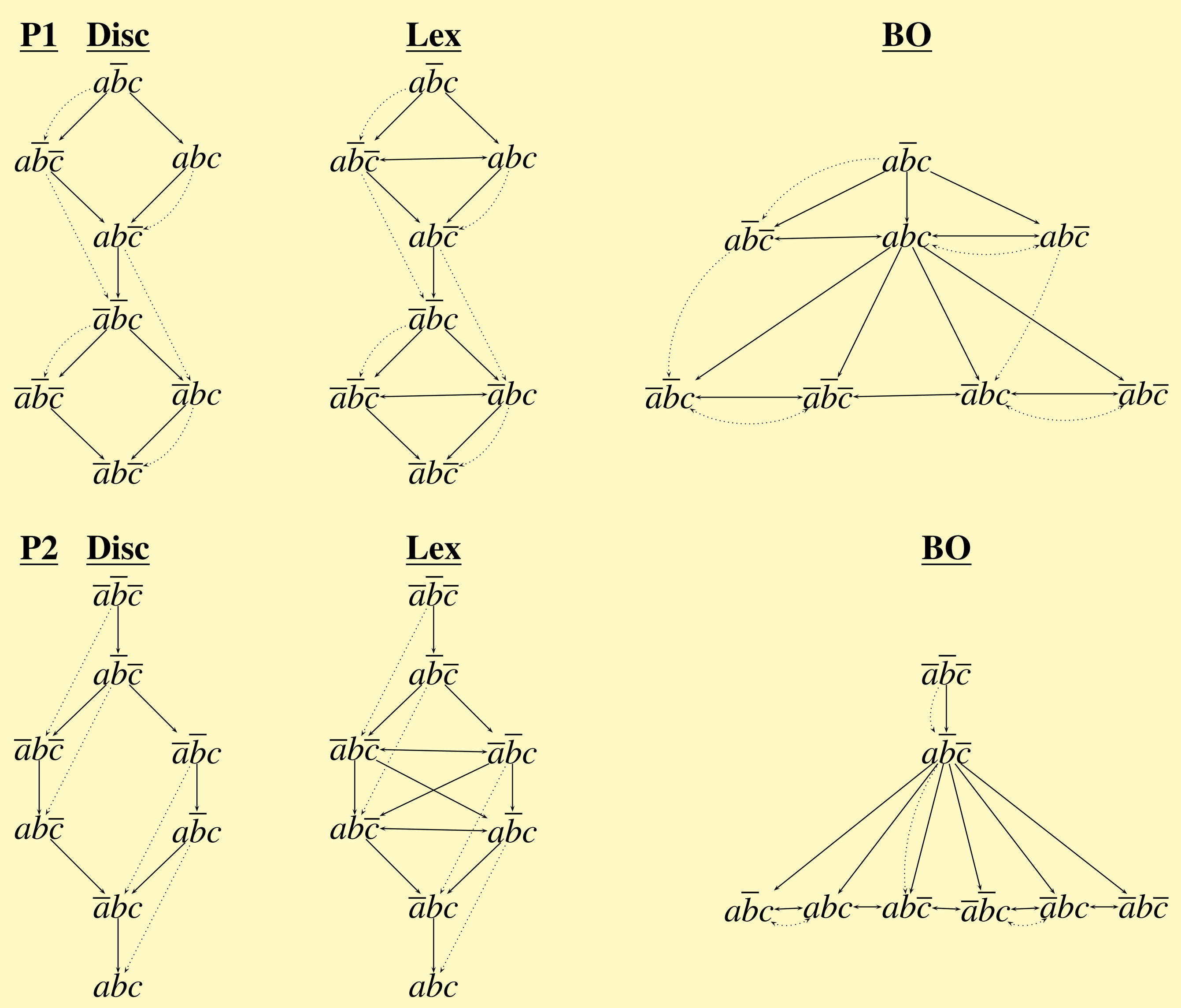
Best-out preference relation Let $a(s) = \min\{j$ such that $\exists \varphi \in \Sigma^j, S \not\models \varphi\}$, with the convention $min(\varnothing) = +\infty$. Then $S \succeq_i^{bo} S'$ iff $a_i(S) \geq a_i(S')$.

A **PG-Boolean game** is a 4-uple $G = (A, V, \pi, \Phi)$, where $\Phi = (\Sigma_1, \ldots, \Sigma_n)$.

- $NE^{disc}_{strong}(G) \subseteq NE^{lex}_{strong}(G) \subseteq NE^{bo}_{strong}(G)$,
- $NE^{lex}_{weak}(G) \subseteq NE^{disc}_{weak}(G) \subseteq NE^{bo}_{weak}(G)$.

$G^{[1 \to k]} = (A, V, \pi, \Phi^{[1 \to k]})$ denotes the k**-reduced game of** G in which all players' goals in G are reduced in their k first strata: $\Phi^{[1 \to k]} = \langle \Sigma_1^{[1 \to k]}, \ldots, \Sigma_n^{[1 \to k]} \rangle$.
Let $c \in \{discr, lex, bo\}$. If S is a SPNE (resp. WPNE) for $Pref^c_{G^{[1 \to k]}}$ of the game $G^{[1 \to k]}$, then S is a SPNE (resp. WPNE) for $Pref^c_{G^{[1 \to (k-1)]}}$ of the game $G^{[1 \to (k-1)]}$.

Let $G = (A, V, \pi, \Phi)$ with $A = \{1, 2\}$, $V = \{a, b, c\}$, $\pi_1 = \{a, c\}$, $\pi_2 = \{b\}$, $\Sigma_1 = \langle a; (\neg b, c) \rangle$, $\Sigma_2 = \langle (\neg b, \neg c); \neg a \rangle$.



- **Discrimin and Leximin**: $NE^{disc}_{weak}(G) = NE^{disc}_{strong}(G) = \{a\bar{b}c\}$
- **Best Out**: $NE^{bo}_{weak}(G) = NE^{bo}_{strong}(G) = \{abc, a\bar{b}c\}$

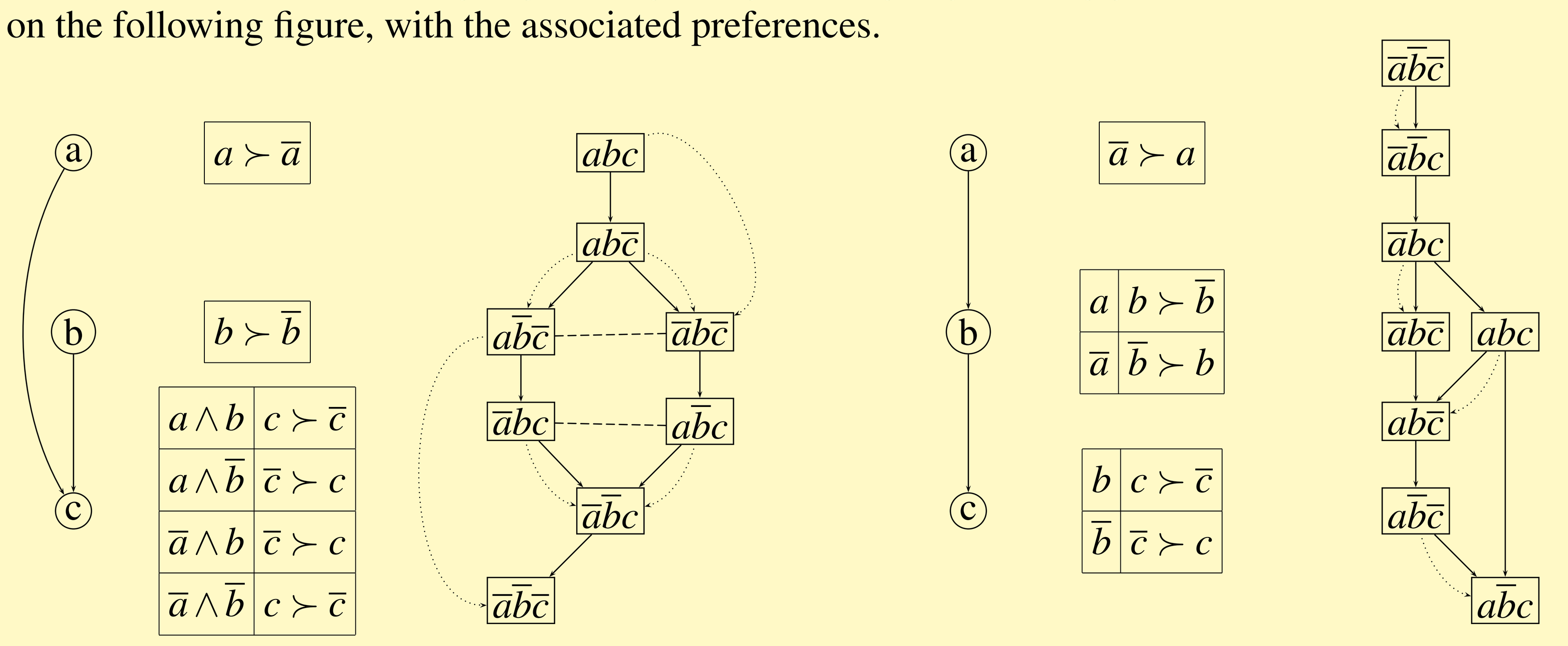
CP-nets

$\mathcal{N} = \langle \mathcal{G}, \mathcal{T} \rangle$ is a **CP-net on** V, where $\mathcal{G}$ is a directed graph over V, and $\mathcal{T}$ is a set of conditional preference tables $CPT(X_j)$ for each $X_j \in V$.
Each $CPT(X_j)$ associates a total order $\succ_p^j$ with each instantiation $p \in 2^{Pa(X_j)}$.
A **CP-boolean game** is a 4-uple $G = (A, V, \pi, \Phi)$, where $\Phi = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$. Each $\mathcal{N}_i$ is a CP-net on V.
Let $G = (A, V, \pi, \Phi)$ be a CP-boolean game such the graphs $\mathcal{G}_i$ are all identical ($\forall i, j$, $\mathcal{G}_i = \mathcal{G}_j$) and acyclic. Then G has one and only one strong PNE.
For each player i, $\mathcal{G}_i$ is denoted by (V, Arc_i), with Arc_i being the set of edges of i's CP-net.

- The **union graph** of G is defined by $\mathcal{G} = (V, Arc_1 \cup \ldots \cup Arc_n)$.
- The **normalized game equivalent to** G, denoted by $G^* = \{A, V, \pi, \Phi^*\}$, is the game obtained from G by rewriting, where
  - the graph of each player's CP-net has been replaced by the graph of the union of CP-nets of G
  - and the CPT of each player's CP-net are modified in order to fit with the new graph, keeping the same preferences

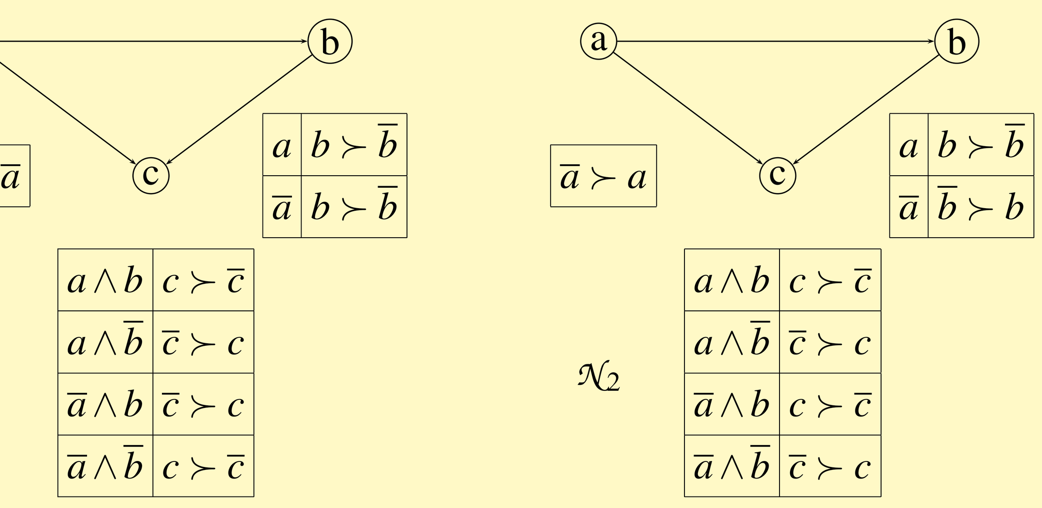
Let $G = (A, V, \pi, \Phi)$ be a CP-boolean game. If the union graph of G is acyclic then G has one and only one SPNE.

$G = (A, V, \pi, \Phi)$ where $A = \{1, 2\}$, $V = \{a, b, c\}$, $\pi_1 = \{a, b\}$, $\pi_2 = \{c\}$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are represented on the following figure, with the associated preferences.



$\mathcal{N}_1$ $\mathcal{N}_2$

Using these partial pre-orders, Nash equilibria are: $NE_{strong} = NE_{weak} = \{abc\}$. It is possible to verify then the union graph is acyclic.



$\mathcal{N}_1$ $\mathcal{N}_2$