# Efficient coalitions in Boolean games

Elise Bonzon, Marie-Christine Lagasquie-Schiex, Jérôme Lang

IRIT, UPS
118 route de Narbonne
F-31062 Toulouse Cédex 9, France
{bonzon,lagasq,lang}@irit.fr

**Abstract**

Boolean games are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. A Boolean game consists of a set of players, each of whom controls a set of propositional variables and has a specific goal expressed by a propositional formula. We show here that Boolean games are a very simple setting, yet sophisticated enough, for studying coalitions. Due to the fact that players have dichotomous preferences, the following notion emerges naturally: a coalition in a Boolean game is efficient if it guarantees that the goal of each member of the coalition is satisfied. We study the properties of efficient coalitions, and we give a characterization of efficient coalitions.

## 1 Introduction

Boolean games [10, 9, 8, 7] are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. Informally, a Boolean game consists of a set of players, each of whom controls a set of propositional variables and has a specific goal expressed by a propositional formula[1]. Thus, a player in a Boolean game has a dichotomous preference relation: either her goal is satisfied or it is not. This restriction may appear at first glance unreasonable. However, many concrete situations can be modelled as games

[1]We refer here to the version of Boolean games defined in [7], that generalizes the initial proposal [10].

where agents have dichotomous preferences (we give such an example in the paper). Moreover, due to the fact that players have dichotomous preferences, the following simple (yet sophisticated enough) notion emerges naturally: a coalition in a Boolean game is efficient if it guarantees that all goals of the members of the coalition are satisfied. Our aim in the following is to define and characterize efficient coalitions, and see how they are related to the well-known concept of core.

After recalling some background of Boolean games in Section 2, we study in Section 3 the properties of effectivity functions associated with Boolean games. In Section 4 we study in detail the notion of efficient coalitions. We give an exact characterization of sets of coalitions that can be obtained as the set of efficient coalitions associated with a Boolean game, and we relate coalition efficiency to the notion of core. Related work and further issues are discussed in Section 5.

## 2 *n*-player Boolean games

For any finite set $V = \{a,b,\dots\}$ of propositional variables, $L_V$ denotes the propositional language built up from $V$, the Boolean constants $\top$ and $\bot$, and the usual connectives. Formulas of $L_V$ are denoted by $\varphi,\psi$ etc. A *literal* is a variable $x$ of $V$ or the negation of a literal. A *term* is a consistent conjunction of literals. A *clause* is a disjunction of literals. If $\varphi \in L_V$, then $Var(\varphi)$ (resp. $Lit(\alpha)$) denotes the set of propositional variables (resp. literals) appearing in $\varphi$.

$2^V$ is the set of the interpretations for $V$, with the usual convention that for $M \in 2^V$ and $x \in V$, $M$ gives the value *true* to $x$ if $x \in M$ and *false* otherwise. $\models$ denotes the consequence relation of classical propositional logic. Let $V' \subseteq V$. A $V'$-interpretation is a truth assignement to each variable of $V'$, that is, an element of $2^{V'}$. $V'$-interpretations are denoted by listing all variables of $V'$, with a $\bar{}$ symbol when the variable is set to false: for instance, let $V' = \{a,b,d\}$, then the $V'$-interpretation $M = \{a,d\}$ assigning $a$ and $d$ to true and $b$ to false is denoted by $a\bar{b}d$. If $Var(\varphi) \subseteq X$, then $Mod_X(\varphi)$ represents the set of $X$-interpretations satisfying $\varphi$.

If $\{V_1,\dots,V_p\}$ is a partition of $V$ and $\{M_1,\dots,M_p\}$ are partial interpretations, where $M_i \in 2^{V_i}$, $(M_1,\dots,M_p)$ denotes the interpretation $M_1 \cup \dots \cup M_p$.

Given a set of propositional variables $V$, a Boolean game on $V$ is an $n$-player game[2],

[2]In the original proposal [10], Boolean games are two-players zero-sum games. However the model

where the actions available to each player consist in assigning a truth value to each variable in a given subset of $V$. The preferences of each player $i$ are represented by a propositional formula $\varphi_i$ formed using the variables in $V$.

**Definition 1.** An $n$**-player Boolean game** is a 5-tuple $(N,V,\pi,\Gamma,\Phi)$, where

- $N = \{1,2,\dots,n\}$ is a set of players (also called agents);
- $V$ is a set of propositional variables;
- $\pi : V \to N$ is a control assignment function;
- $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ is a set of constraints, where each $\gamma_i$ is a satisfiable propositional formula of $L_{\pi(i)}$;
- $\Phi = \{\varphi_1,\dots,\varphi_n\}$ is a set of goals, where each $\varphi_i$ is a satisfiable formula of $L_V$.

A 4-tuple $(N,V,\pi,\Gamma)$, with $N,V,\pi,\Gamma$ defined as above, is called a **pre-Boolean game**.

The control assignment function $\pi$ maps each variable to the player who controls it. For ease of notation, the set of all the variables controlled by a player $i$ is written $\pi_i$ such as $\pi_i = \{x \in V | \pi(x) = i\}$. Each variable is controlled by one and only one agent, that is, $\{\pi_1,\dots,\pi_n\}$ forms a partition of $V$.
For each $i$, $\gamma_i$ is a constraint restricting the possible strategy profiles for player $i$.

**Definition 2.** Let $G = (N,V,\pi,\Gamma,\Phi)$ be a Boolean game. A **strategy**[3] for player $i$ in $G$ is a $\pi_i$-interpretation satisfying $\gamma_i$. The set of strategies for player $i$ in $G$ is $S_i = \{s_i \in 2^{\pi_i} \mid s_i \models \gamma_i\}$. A **strategy profile** $s$ for $G$ is a $n$-tuple $s = (s_1,s_2,\dots,s_n)$ where for all $i$, $s_i \in S_i$. $S = S_1 \times \dots \times S_n$ is the set of all strategy profiles.

Note that since $\{\pi_1,\dots,\pi_n\}$ forms a partition of $V$, a strategy profile $s$ is an interpretation for $V$, i.e., $s \in 2^V$. The following notations are usual in game theory. Let $s = (s_1,\dots,s_n)$ be a strategy profile. For any nonempty set of players $I \subseteq N$, the projection of $s$ on $I$ is defined by $s_I = (s_i)_{i\in I}$ and $s_{-I} = s_{N\setminus I}$. If $I = \{i\}$, we denote the projection of $s$ on $\{i\}$ by $s_i$ instead of $s_{\{i\}}$; similarly, we note $s_{-i}$ instead of $s_{-\{i\}}$. $\pi_I$ denotes the set of the variables controlled by $I$, and $\pi_{-I} = \pi_{N\setminus I}$. The set of strategies for $I \subseteq N$ is $S_I = \times_{i\in I} S_i$, and the set of goals for $I \subseteq N$ is $\Phi_I = \bigwedge_{i\in I} \varphi_i$.

can easily be generalized to $n$ players and non necessarily zero-sum games [7].

[3] In this paper, only pure strategies are considered.

If $s$ and $s'$ are two strategy profiles, $(s_{-I}, s'_I)$ denotes the strategy profile obtained from $s$ by replacing $s_i$ with $s'_i$ for all $i \in I$.
The goal $\varphi_i$ of player $i$ is a compact representation of a dichotomous preference relation, or equivalently, of a binary utility function $u_i : S \to \{0,1\}$ defined by $u_i(s) = 0$ if $s \models \neg\varphi_i$ and $u_i(s) = 1$ if $s \models \varphi_i$. $s$ is at least as good as $s'$ for $i$, denoted by $s \succeq_i s'$, if $u_i(s) \geq u_i(s')$, or equivalently, if $s \models \neg\varphi_i$ implies $s' \models \neg\varphi_i$; $s$ is strictly better than $s'$ for $i$, denoted by $s \succ_i s'$, if $u_i(s) > u_i(s')$, or, equivalently, $s \models \varphi_i$ and $s' \models \neg\varphi_i$.
Note that this choice of binary utilites clearly implies a loss of generality. However, some interesting problems, as in Example 2, have preferences that are naturally dichotomous, and Boolean games allow to represent these problems in a compact way. Furthermore, Boolean games can easily be extended so as to allow for non-dichotomous preferences, represented in some compact language for preference representation (see [5]).

## 3 Coalitions and effectivity functions in Boolean games

Effectivity functions have been developed in social choice to model the ability of coalitions [11, 1, 14]. As usual, a **coalition** $C$ is any subset of $N$. $N$ is called the **grand coalition**. Given a set of alternatives $S$ from which a set of agents $N$ have to choose, an effectivity function $\mathrm{Eff}\colon 2^N \to 2^{2^S}$ associates a set of subsets of $S$ with each coalition. $X \in \mathrm{Eff}(C)$ is interpreted as "coalition $C$ is effective for $X$".

**Definition 3.** A **coalitional effectivity function** is a function $\mathrm{Eff}\colon 2^N \to 2^{2^S}$ which is monotonic: for every coalition $C \subseteq N$, $X \in \mathrm{Eff}(C)$ implies $Y \in \mathrm{Eff}(C)$ whenever $X \subseteq Y \subseteq S$.

The function Eff associates to every group of players the set of outcomes for which the group is effective. We usually interpret $X \in \mathrm{Eff}(C)$ as "the players in $C$ have a joint strategy for bringing about an outcome in $X$".
In [14], the meaning of "effective" is precised in the framework of strategic games by defining "$\alpha$-effectivity": a coalition $C \subseteq N$ is **$\alpha$-effective** for $X \subseteq S$ if and only if players in $C$ have a joint strategy to achieve an outcome in $X$ *no matter* what strategies the other players choose.

As Boolean games are a specific case of strategic games, we would like to define $\alpha$-

effectivity functions in this framework. One of the features of Boolean games is the definition of individual strategies as truth assignments of a given set of propositional variables. We might wonder how restrictive this specificity is. In this section we study Boolean games from the point of view of effectivity functions. Clearly, the definition of $S_i$ as $Mod_{\pi_i}(\gamma_i)$ induces some constraints on the power of players. Our aim is to give an exact characterization of $\alpha$-effectivity functions induced by Boolean games. Since in Boolean games the power of an agent $i$ is her goal $\varphi_i$, it suffices to consider pre-Boolean games only when dealing with effectivity functions. A pre-Boolean game $G$ induces an $\alpha$-effectivity function $\mathrm{Eff}_G$ as follows:

**Definition 4.** Let $G = (N,V,\pi,\Gamma)$ be a pre-Boolean game. The coalitional $\alpha$-effectivity function induced by $G$ is the function $\mathrm{Eff}_G : 2^N \to 2^{2^S}$ defined by: for any $X \subseteq S$ and any $C \subseteq N$, $X \in \mathrm{Eff}_G(C)$ if there exists $s_C \in S_C$ such that for any $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \in X$. [4]

This definition is a particular case of the $\alpha$-effectivity function induced by a strategic game (see [14], chapter 2). Therefore, these functions satisfy the following properties (cf. [14], Theorem 2.27): (i) $\forall C \subseteq N$, $\varnothing \notin \mathrm{Eff}(C)$; (ii) $\forall C \subseteq N$, $S \in \mathrm{Eff}(C)$; (iii) for all $X \subseteq S$, if $\overline{X} \notin \mathrm{Eff}(\varnothing)$ then $X \in \mathrm{Eff}(N)$; (iv) Eff is superadditive, that is, if for all $C, C' \subseteq N$ and $X, Y \subseteq S$, $X \in \mathrm{Eff}(C)$ and $Y \in \mathrm{Eff}(C')$, then $X \cap Y \in \mathrm{Eff}(C \cup C')$. An effectivity function satisfying these four properties is called **strongly playable**. Note that strong playability implies regularity and coalition-monotonicity ([14], Lemma 2.26).

However, pre-Boolean games are a specific case of strategic game forms, therefore we would like to have an exact characterization of those effectivity functions that correspond to a pre-Boolean game. We first have to define two additional properties. Define $At(C)$ as the minimal sets in $\mathrm{Eff}(C)$, that is, $At(C) = \{X \in \mathrm{Eff}(C) \mid$ there is no $Y \in \mathrm{Eff}(C)$ such that $Y \subseteq X\}$.

**Atomicity:** Eff satisfies *atomicity* if for every $C \subseteq N$, $At(C)$ forms a partition of $S$.

[4] Note that effectivity functions induced by pre-Boolean games may be equivalently expressed as mappings $\mathrm{Eff} : 2^N \to 2^{L_V}$ from coalitions to sets of logical formulas: $\varphi \in \mathrm{Eff}(I)$ if $Mod_{\pi_I}(\varphi) \in \mathrm{Eff}(I)$. This definition obviously implies syntax-independence, that is, if $\varphi \equiv \psi$ then $\varphi \in \mathrm{Eff}(I)$ iff $\psi \in \mathrm{Eff}(I)$.

**Decomposability:** Eff satisfies *decomposability* if for every $I, J \subseteq N$ and for every $X \subseteq S$, $X \in \mathrm{Eff}(I \cup J)$ if and only if there exist $Y \in \mathrm{Eff}(I)$ and $Z \in \mathrm{Eff}(J)$ such that $X = Y \cap Z$.

Note that decomposability is a strong property that implies superadditivity.

**Proposition 1.** A coalitional effectivity function Eff satisfies (1) strong playability, (2) atomicity, (3) decomposability and (4) $\mathrm{Eff}(N) = 2^S \setminus \varnothing$ if and only if there exists a pre-Boolean game $G = (N, V, \pi, \Gamma)$ and an injective function $\mu : S \to 2^V$ such that for every $C \subseteq N$: $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$.

*Sketch of proof:* [5] The ($\Leftarrow$) direction does not present any difficulty: we can easily prove than $\mathrm{Eff}_G$ satisfies strong playability (from Theorem 2.27 in [14]), atomicity, decomposability and $\mathrm{Eff}_G(N) = 2^S \setminus \varnothing$. As $\mu$ is a bijection between $S$ and $\mu(S)$, these properties transfer to Eff.

For the ($\Rightarrow$) direction, we first show than for every $s \in S$, there exists a unique $(Z_1, \ldots, Z_n)$ such that for every $i$, $Z_i \in At(i)$ and $Z_1 \cap \ldots \cap Z_n = \{s\}$. Then, we build $G$ from Eff as follows:

- for every $i$, number $At(i)$: let $r_i$ be a bijective mapping from $At(i)$ to $\{0, 1, \ldots, |At(i)| - 1\}$. Then create $p_i = \lceil \log_2 |At(i)| \rceil$ propositional variables $x_i^1, \ldots, x_i^{p_i}$. Finally, let $V = \{x_i^j | i \in N, 1 \le j \le p_i\}$;
- for each $i$: $\pi_i = \{x_i^1, \ldots, x_i^{p_i}\}$;
- for each $i$ and each $j \le p_i$, let $\varepsilon_{i,j}$ be the $j$th digit in the binary representation of $p_i$. Note that $\varepsilon_{i,p_i} = 1$ by definition of $p_i$. If $x$ is a propositional variable then we use the following notation: $0.x = \neg x$ and $1.x = x$. Then define $\gamma_i = \bigwedge_{j \in \{2,\ldots,p_i\}, \varepsilon_{i,j}=0} \left( \bigwedge_{1 \le k \le j-1} \varepsilon_{i,j}.x_i^k \to \neg x_i^j \right)$
- finally, for each $s \in S$, let $\mu(s) \in 2^V$ defined by: $x_i^j \in \mu(s)$ if and only if the $j$th digit of the binary representation of $r_i(Z_i(s))$ is 1.

For every $i \in N$ and every $Z \in At(i)$, let $k = r_i(Z)$ and $s_i(Z)$ the strategy of player in $i$ in $G$ corresponding to the binary representation of $k$ using $\{x_i 1, \ldots, x_i^{p_i}\}$, $x_i 1$ being the most significant bit. For instance, if $p_i = 3$ and $r_i(Z_i) = 6$ then $s_i(Z) = (x_i 1, x_i 2, \neg x_i 3)$.

[5] A complete version of this proof can be found in [6].

**Note:** To follow the proof, it may be helpful to see how this construction works on an example. Let $N = \{1,2,3\}$, $S = \{1,2,3,4,5,6,7,8,9,A,B,C\}$, $At(1) = \{1234, 5678, 9ABC\}$, $At(2) = \{13579B, 2468AC\}$, $At(3) = \{12569C, 3478AB\}$ (parentheses for subsets of $S$ are omitted – 1234 means $\{1,2,3,4\}$ and so on). By decomposability,we have $At(12) = \{13,24,57,68,9B,AC\}$, $At(13) = \{12,34,56,78,9C,AB\}$, and $At(23) = \{159,37B,26C,48A\}$. $|At(1)| = 3$, therefore $p_1 = 2$. $|At(2)| = |At(3)| = 2$, therefore $p_2 = p_3 = 1$. Thus, $V = \{x_1 1, x_1 2, x_2 1, x_3 1\}$. Let $At(1) = \{Z_0, Z_1, Z_2\}$, that is, $r_1(1234) = 0$, $r_1(5678) = 1$ and $r_1(9ABC) = 2$. Likewise, $r_2(13579B) = 0$, $r_2(2468AC) = 1$, $r_3(12569C) = 0$ and $r_3(3478AB) = 1$. Consider $s = 6$. We have $s = 5678 \cap 2468AC \cap 12569C$, therefore $s_G = \mu(s) = (\neg x_1 1, x_1 2, x_2 1, \neg x_3 1)$. The constraints are $\gamma_1 = (x_1 1 \to \neg x_1 2)$, $\gamma_2 = \gamma_3 = \top$.
Then, we show that for every $C$, $\mathrm{Eff}_G(C) = \mu(\mathrm{Eff}(C))$. The proof, though rather long, does not present any particular difficulty. See [6]. ■

## 4 Efficient coalitions

### 4.1 Definitions and characterization

We now consider full Boolean games and define *efficient coalitions*. Informally, a coalition is efficient in a Boolean game if and only if it has the ability to jointly satisfy the goals of all members of the coalition:

**Definition 5.** Let $G = (N,V,\pi,\Gamma,\Phi)$ be a Boolean game. A coalition $C \subseteq N$ is **efficient** if and only if $\exists s_C \in S_C$ such that $\forall s_{-C}$, $s_C \models \bigwedge_{i \in C} \varphi_i$. The set of all efficient coalitions of a game $G$ is denoted by $\mathrm{EC}(G)$.

**Example 1.** Let $G = (N,V,\Gamma,\pi,\Phi)$ where $V = \{a,b,c\}$, $N = \{1,2,3\}$, $\gamma_i = \top$ for every $i$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\varphi_1 = (\neg a \wedge b)$, $\varphi_2 = (\neg a \vee \neg c)$ and $\varphi_3 = (\neg b \wedge \neg c)$.
Observe first that $\varphi_1 \wedge \varphi_3$ is inconsistent, therefore no coalition containing $\{1,3\}$ can be efficient. $\{1\}$ is not efficient, because $\varphi_1$ cannot be made true only by fixing the value of $a$; similarly, $\{2\}$ and $\{3\}$ are not efficient either. $\{1,2\}$ is efficient, because the joint strategy $s_{\{1,2\}} = \overline{a}b$ is such that $s_{\{1,2\}} \models \varphi_1 \wedge \varphi_2$. $\{2,3\}$ is efficient, because $s_{\{2,3\}} = \overline{b}\overline{c} \models \varphi_2 \wedge \varphi_3$. Obviously, $\varnothing$ is efficient[6], because $\varphi_\varnothing = \bigwedge_{i \in \varnothing} \varphi_i \equiv \top$ is

[6]One may argue this makes little sense to say that the empty coalition is efficient. Anyway, the defi-

always satisfied. Therefore, $\mathrm{EC}(G) = \{\varnothing, \{1,2\}, \{2,3\}\}$.

From this simple example we see already that EC is neither downward closed nor upward closed, that is, if $C$ is efficient, then a subset or a superset of $C$ may not be efficient. We also see that EC is not closed under union or intersection: $\{1,2\}$ and $\{2,3\}$ are efficient, but neither $\{1,2\} \cap \{2,3\}$ nor $\{1,2\} \cup \{2,3\}$ is.

**Example 2** (kidney exchange, after [2])**.** Consider $n$ pairs of individuals, each consisting of a recipient $R_i$ in urgent need of a kidney transplant, and a donor $D_i$ who is ready to give one of her kidneys to save $R_i$. As $D_i$'s donor kidney is not necessarily compatible with $R_i$, a strategy for saving more people consists in considering the graph $\langle\{1,\ldots,n\},E\rangle$ containing a node $i \in 1,\ldots,n$ for each pair $(D_i,R_i)$ and containing the edge $(i,j)$ whenever $D_i$'s kidney is compatible with $R_j$. A solution is any set of nodes that can be partitioned into disjoint cycles in the graph: in a solution, a donor $D_i$ gives a kidney if and only if $R_i$ is given one. An optimal solution (saving a maximum number of lifes) is a solution with a maximum number of nodes. The problem can be seen as the following Boolean game $G$:

- $N = \{1,\ldots,n\}$;
- $V = \{g_{ij} | i,j \in \{1,\ldots,n\}\}$; $g_{ij}$ being true means that $D_i$ gives a kidney to $R_j$.
- $\pi_i = \{g_{ij}; 1 \leq j \leq n\}$;
- for every $i$, $\gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$ expresses that a donor cannot give more than one kidney.
- for every $i$, $\varphi_i = \bigvee_{(j,i) \in E} g_{ji}$ expresses that the goal of $i$ is to be given a kidney that is compatible with $R_i$.

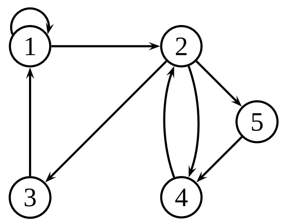
For example, take $n = 5$ and $E = \{(1,1),(1,2),(2,3),(2,4),(2,5),(3,1),(4,2),(5,4)\}$. Then $G = (N,V,\Gamma,\pi,\Phi)$, with

- $N = \{1,2,3,4,5\}$
- $V = \{g_{ij} \mid 1 \leq i,j \leq 5\}$;
- $\forall i$, $\gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$
- $\pi_1 = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\}$, and similarly for $\pi_2$, etc.

nition of an efficient coalition could be changed so as to exclude $\varnothing$, further notions and results would be unchanged.

- $\varphi_1 = g_{11} \vee g_{31}$; $\varphi_2 = g_{12} \vee g_{42}$; $\varphi_3 = g_{23}$; $\varphi_4 = g_{24} \vee g_{54}$; $\varphi_5 = g_{25}$.

The corresponding graph is depicted below.




Clearly enough, efficient coalitions correspond to solutions. In our example, the efficient coalitions are $\varnothing$, $\{1\}$, $\{2,4\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{2,4,5\}$ and $\{1,2,4,5\}$.

We have seen that the set of efficient coalitions associated with a Boolean game may not be downward closed nor upward closed, nor closed under union or non-empty intersection. We find that it is possible to characterize the efficient coalitions of a Boolean game.

**Proposition 2.** Let $N = \{1,\ldots,n\}$ be a set of agents and $\mathcal{SC} \in 2^{2^N}$ a set of coalitions. There exists a Boolean game $G$ over $N$ such that the set of efficient coalitions for $G$ is $\mathcal{SC}$ (i.e. $\mathrm{EC}(G) = \mathcal{SC}$) if and only if $\mathcal{SC}$ satisfies these two properties:

**(1)** $\varnothing \in \mathcal{SC}$.

**(2)** for all $I,J \in \mathcal{SC}$ such that $I \cap J = \varnothing$, $I \cup J \in \mathcal{SC}$.

Thus, a set of coalitions corresponds to the set of efficient coalitions for some Boolean game if and only if (a) it contains the empty set and (b) it is closed by union of disjoint coalitions.

*Sketch of proof:* [7] The ($\Rightarrow$) direction is proven easily; intuitively, when two disjoint coalitions $I$ and $J$ are efficient, each one has a strategy guaranteeing its goals to be satisfied, and the joint strategies of $I$ and $J$ guarantee that the goals of all agents in $I \cup J$ are satisfied. As seen in Example 1, this is no longer true when $I$ and $J$ intersect. The ($\Leftarrow$) direction of the proof is more involved and needs the following Boolean game $G$ to be constructed for each set of coalitions $\mathcal{SC}$ satisfying (1) and (2):

[7] A complete version of this proof can be found in [6].

- $V = \{connect(i,j) | i,j \in N\}$ (all possible connections between players);
- $\forall i, \gamma_i = \top$;
- $\pi_i = \{connect(i,j) | j \in N\}$ (all connections from player $i$);
- $\varphi_i = \bigvee_{I \in \mathcal{SC}, i \in I} F_I$,
  where $F_I = \left(\bigwedge_{j,k \in I} connect(j,k)\right) \wedge \left(\bigwedge_{j \in I, k \notin I} \neg connect(j,k)\right)$ (player $i$ wants all the players of her coalition to be connected with each other and disconnected from the players outside the coalition).

We want to show that $\mathrm{EC}_G = \mathcal{SC}$ (where $\mathrm{EC}_G$ is the set of efficient coalitions for $G$). We first show that $\mathcal{SC} \subseteq \mathrm{EC}_G$. Let $I \in \mathcal{SC}$. If every agent $i \in I$ plays $\left(\bigwedge_{j \in I} connect(i,j)\right) \wedge \left(\bigwedge_{k \notin I} \neg connect(i,k)\right)$, then $\varphi_i$ is satisfied for every $i \in I$. Hence, $I$ is an efficient coalition for $G$ and $\mathcal{SC}$ is included in $\mathrm{EC}(G)$.

In order to prove that $\mathrm{EC}_G \subseteq \mathcal{SC}$, we define a *covering of a coalition I by disjoint subsets of* $\mathcal{SC}$ as a tuple $\vec{C} = \langle C_i | i \in I \rangle$ of coalitions such that: (a) for every $k \in I$, $C_k \in \mathcal{SC}$; (b) for all $C_j, C_k \in \vec{C}$, either $C_j = C_k$ or $C_j \cap C_k = \varnothing$; (c) for every $i \in I$, $i \in C_i$. Let $Cov(\mathcal{SC}, I)$ be the set of all covering of $I$ by disjoint subsets of $\mathcal{SC}$.
For instance, if $\mathcal{SC} = \{\varnothing, 1, 24, 123, 124\}$ then $Cov(\mathcal{SC}, 12) = \{\langle 1, 24 \rangle, \langle 123, 123 \rangle, \langle 124, 124 \rangle\}$[8], $Cov(\mathcal{SC}, 124) = \{\langle 1,1,24 \rangle, \langle 1,24,24 \rangle, \langle 124,124,124 \rangle\}$, $Cov(\mathcal{SC}, 123) = \{\langle 123,123,123 \rangle\}$ and $Cov(\mathcal{SC}, 234) = Cov(\mathcal{SC}, 1234) = \varnothing$.

The proof goes along the following steps:

**L1** For any collection $Col = \{C_i, i = 1, \ldots, q\} \subseteq 2^{2^N}$, $\bigwedge_{1 \le i \le q} F_{C_i}$ is satisfiable if and only if for any $i, j \in \{1, \ldots, q\}$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.

**L2** From L1, we deduce that $\forall I \neq \varnothing$, $\Phi_I$ is equivalent to $\bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$.

**L3** From property (2) (assumption of Proposition 2) and L2, we can prove that if $I \subseteq 2^N$, then $\Phi_I$ is satisfiable if and only if there exists $J \in \mathcal{SC}$ such that $I \subseteq J$.

Let $I$ be an efficient coalition such that $I \notin \mathcal{SC}$ (which implies $I \neq \varnothing$, because by assumption $\varnothing \in \mathcal{SC}$).

- If $I = N$ then there is no $J \in \mathcal{SC}$ such that $I \subseteq J$ (because $I \notin \mathcal{SC}$), and then L3 implies that $\Phi_I$ is unsatisfiable, therefore $I$ cannot be efficient for $G$.

[8] There are two players in $I = \{1, 2\}$, therefore each $\vec{C}$ in $Cov(\mathcal{SC}, 12)$ contains 2 coalitions, one for each player, satisfying (a), (b) and (c).

- Assume now that $I \neq N$ and define the following $\overline{I}$-strategy $S_{\overline{I}}$ ($\overline{I} = N \setminus I$): for every $i \in \overline{I}$, $s_i = \{\neg connect(i,j) | j \in I\}$ (plus whatever on the variables $connect(i,j)$ such that $j \notin I$). Let $\vec{C} = \langle C_i, i \in I\rangle \in Cov(\mathcal{SC}, I)$.

  We first claim that there is a $i^* \in I$ such that $C_{i^*}$ is not contained in $I$. Indeed, suppose that for every $i \in I$, $C_i \subseteq I$. Then, because $i \in C_i$ holds for every $i$, we have $\bigcup_{i\in I} C_i = I$. Now, $C_i \in \mathcal{SC}$ for all $i$, and any two distinct $C_i, C_j$ are disjoint, therefore, by property (2) we get $I \in \mathcal{SC}$, which by assumption is false.

  Now, let $k \in C_{i^*} \setminus I$ (such a $k$ exists because $C_{i^*}$ is not contained in $I$). Now, the satisfaction of $F_{C_i}$ requires $connect(k, i^*)$ to be true, because both $i$ and $k$ are in $C_i$. Therefore $s_k \models \neg F_{C_i}$, and a fortiori $s_{\overline{I}} \models \neg F_{C_i}$, which entails $s_{\overline{I}} \models \neg \bigwedge_{i\in I} F_{C_i}$.

  This being true for any $\vec{C} \in Cov(\mathcal{SC}, I)$, we have $s_{\overline{I}} \models \bigwedge_{\vec{C}\in Cov(\mathcal{SC},I)} \neg \bigwedge_{i\in I} F_{C_i}$. that is, $s_{\overline{I}} \models \neg \bigvee_{\vec{C}\in Cov(\mathcal{SC},I)} \bigwedge_{i\in I} F_{C_i}$. Together with L2, this entails $s_{\overline{I}} \models \neg\Phi_I$. Hence, $I$ does not control $\Phi_I$ and $I$ cannot be efficient for $G$.

■

The notion of efficient coalition is the same as the notion of successful coalition in qualitative coalitional games (QCG) introduced in [16], even if, as we discuss in Section 5, QCG and Boolean games are quite different.

### 4.2 Efficient coalitions and the core

We now relate the notion of efficient coalition to the usual notion of core of a coalitional game. In coalitional games with ordinal preferences, the core is usually defined as follows (see e.g. [4, 13, 12]): a strategy profile $s$ is in the core of a coalitional game if and only if there exists no coalition $C$ with a joint strategy $s_C$ that guarantees that all members of $C$ are better off than with $s$. Here we consider also a stronger notion of core: a strategy profile $s$ is in the strong core of a coalitional game if and only if there exists no coalition $C$ with a joint strategy $s_C$ that guarantees that all members of $C$ are at least as satisfied as with $s$, and at least one member of $C$ is strictly better off than with $s$.

**Definition 6.** Let $G$ be a Boolean game.
The **(weak) core** of $G$, denoted by $WCore(G)$, is the set of strategy profiles $s =$

$(s_1,\ldots,s_n)$ such that there exists no $C \subset N$ and no $s_C \in S_C$ such that for every $i \in C$ and every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succ_i s$.
The **strong core** of a Boolean game $G$, denoted by $SCore(G)$, is the set of strategy profiles $s = (s_1,\ldots,s_n)$ such that there exists no $C \subset N$ and no $s_C \in S_C$ such that for every $i \in C$ and every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succeq_i s$ and there is an $i \in C$ such that for every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succ_i s$.

This concept of weak core is equivalent[9] to the notion of **strong Nash equilibrium** introduced by [3], where coalitions form in order to correlate the strategies of their members. This notion involves, at least implicitly, the assumption that cooperation necessarily requires that players be able to sign “binding agreements”: players have to follow the strategies they have agreed upon, even if some of them, in turn, might profit by deviating. However, if players of a coalition $C$ agreed for a strategy $s_C$, at least one player $i \in C$ is satisfied by this strategy: we have $\exists i \in C$ such that $s \models \varphi_i$.

The relationship between the (weak) core of a Boolean game and its set of efficient coalitions is expressed by the following simple result. The proofs of following results can be found in [6]:

**Proposition 3.** Let $G = (N,V,\Gamma,\pi,\Phi)$ be a Boolean game. $s \in WCore(G)$ if and only if $s$ satisfies at least one member of every efficient coalition, that is, for every $C \in \mathrm{EC}(G)$, $s \models \bigvee_{i \in C} \varphi_i$.

In particular, when no coalition of a Boolean game $G$ is efficient, then all strategy profiles are in $WCore(G)$. Moreover, the weak core of a Boolean game cannot be empty:

**Proposition 4.** For any Boolean game $G$, $WCore(G) \neq \varnothing$.

The strong core of a Boolean game is harder to characterize in terms of efficient coalitions. We only have the following implication.

**Proposition 5.** Let $G = (N,V,\Gamma,\pi,\Phi)$ be a Boolean game, and $s$ be a strategy profile. If $s \in SCore(G)$ then for every $C \in \mathrm{EC}(G)$ and every $i \in C$, $s \models \varphi_i$.

[9]This equivalence is easily shown: it is just a rewriting of the definition given in [3].

Thus, a strategy in the strong core of $G$ satisfies the goal of every member of every efficient coalition. The following counterexample shows that the converse does not hold.

**Example 3.** Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. We have: $V = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3, 4, 5, 6\}$, $\gamma_i = \top$ for every $i$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\pi_4 = \{d\}$, $\pi_5 = \{e\}$, $\pi_6 = \{f\}$, $\varphi_1 = b \vee d$, $\varphi_2 = a \vee c$, $\varphi_3 = \neg b \vee d$, $\varphi_4 = e$, $\varphi_5 = \neg a \wedge \neg b \wedge \neg c$ and $\varphi_6 = \neg a \wedge \neg c \wedge \neg d$.
This game has two efficient coalitions: $\{1, 2\}$ and $\{2, 3\}$.
Let $s = abcd\overline{e}f$. We have $s \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \neg\varphi_4 \wedge \neg\varphi_5 \wedge \neg\varphi_6$. So, $\forall C \in \mathrm{EC}(G)$, $\forall i \in C$, $s \models \varphi_i$.
However, $s \notin SCore(G)$: $\exists C' = \{1, 2, 3, 4, 5\} \subset N$ such that $\exists s_C = abcde \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \neg\varphi_5$. So, $\forall s_{-C}$, $(s_C, s_{-C}) \succeq_1 s$, $(s_C, s_{-C}) \succeq_2 s$, $(s_C, s_{-C}) \succeq_3 s$, $(s_C, s_{-C}) \succeq_5 s$, and $(s_C, s_{-C}) \succ_4 s$. $s \notin SCore(G)$.

Note that the strong core of a Boolean game can be empty: in Example 1, the set of efficient coalitions is $\{\varnothing, \{1, 2\}, \{2, 3\}\}$, therefore there is no $s \in S$ such that for all $C \in \mathrm{EC}(G)$, for all $i \in C$, $s \models \varphi_i$, therefore, $SCore(G) = \varnothing$. However, we can show than the non-emptyness of the strong core is equivalent to the following simple condition on efficient coalitions.

**Proposition 6.** Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. We have the following: $Score(G) \neq \emptyset$ if and only if $\bigcup\{C \subseteq N | C \in \mathrm{EC}(G)\} \in \mathrm{EC}(G)$, that is, if and only if the union of all efficient coalitions is efficient.

## 5 Conclusion

We have shown that Boolean games can be used as a compact representation setting for coalitional games where players have dichotomous preferences. This specificity lead us to define an interesting notion of efficient coalitions. We have given an exact characterization of sets of coalitions that correspond to the set of efficient coalitions for a Boolean game, and we have given several results concerning the computation of efficient coalitions.
Note that some of our notions and results do not explicitly rely on the use of propositional logic. For instance, efficient coalitions can be defined in a more general

setting where goals are simply expressed as nonempty sets of states. However, many notions (in particular, the control assignment function $\pi$) become much less clear when abstracting from the propositional representation.

Clearly, a limitation of our results is that they apply to dichotomous preferences only. However, as illustrated on Example 2, some problems are naturally expressed with dichotomous goals. Moreover, it is always worth starting by studying simple cases, especially when they already raise complex notions[10].

As Boolean games, qualitative coalitional games (QCG), introduced in [16], are games in which agents are not assigned utility values over outcomes, but are satisfied if their goals are achieved. A first difference between QCG and Boolean games is that there is no control assignment function in QCG. A second one is that each agent in QCG can have a set of goals, and is satisfied if at least one of her goals is satisfied, whereas each agent in Boolean games has a unique goal. However, QCG's characteristic function, which associates to each coalition $C$ the sets of goals that members of $C$ can achieve, corresponds in Boolean games to the set $W(C) = \{X \subseteq \{\varphi_1, \ldots, \varphi_n\}$ such that $\exists s_C \in S_C : s_C \models \varphi_i\}$[11].

Coalition logic [14] allows to express, for any coalition $C$ and any formula $\varphi$, the ability of $C$ to ensure that $\varphi$ hold (which is written $[C]\varphi$). In Boolean games, the power of agents, expressed by the control assignment function $\pi$, is still in the meta-language. Expressing $\pi$ within coalition logic would however be possible, probably using ideas from [15]. The next step would then consist in introducing goals into coalition logic. This is something we plan to do in the near future.

## References

[1] J. Abdou and H. Keiding. *Effectivity functions in social choice*. Kluwer, 1991.

[2] D. Abraham, A. Blum, and T. Sandholm. Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchange. In *Proceedings of EC-2007*, 2007. To

[10] Note that Boolean games can easily be extended with propositional languages for representing compactly nondichotomous preferences [5].

[11] For instance, we have for Example 1 : $W(\{1\}) = W(\{3\}) = W(\{1,3\}) = \{\varphi_2\}$, $W(\{2\}) = \varnothing$, $W(\{1,2\}) = \{\varphi_1, \varphi_2\}$, $W(\{2,3\}) = \{\varphi_2, \varphi_3\}$, $W(\{1,2,3\}) = \{\{\varphi_1, \varphi_2\}, \{\varphi_2, \varphi_3\}\}$.

appear.

[3] R.J. Aumann. Acceptable points in general n-person games. In A.W. Tucker and R.D. Luce, editors, *Contributions to the Theory of Games IV*. Princeton University Press, 1959.

[4] R.J. Aumann. A survey of cooperative games without side payments. In M. Shubik, editor, *Essays in Mathematical Economics in Honour of Oskar Morgenstern*. Princeton University Press, 1967.

[5] E. Bonzon, M.C. Lagasquie-Schiex, and J. Lang. Compact preference representation for boolean games. In *Proc. of PRICAI'06*, volume 4099, pages 41–50. Springer-Verlag, 2006.

[6] E. Bonzon, MC Lagasquie-Schiex, and J. Lang. Efficient coalitions in boolean games. Research report IRIT/RR–2007-13–FR, IRIT, 2007. ftp://ftp.irit.fr/IRIT/RPDMP/report.pdf.

[7] E. Bonzon, M.C. Lagasquie-Schiex, J. Lang, and B. Zanuttini. Boolean games revisited. In *Proc. of ECAI'06*, pages 265–269. Springer-Verlag, 2006.

[8] P.E. Dunne and W. van der Hoek. Representation and complexity in boolean games. In *Proc. of JELIA2004*, volume LNCS 3229, pages 347–359. José Júlio Alferes et João Alexandre Leite (eds), 2004.

[9] P. Harrenstein. *Logic in Conflict*. PhD thesis, Utrecht University, 2004.

[10] P. Harrenstein, W. van der Hoek, J.J Meyer, and C. Witteveen. Boolean games. In J. van Benthem, editor, *Proc. of TARK 2001*, volume Theoretical Aspects of Rationality and Knowledge, pages 287–298. Morgan Kaufmann, 2001.

[11] H. Moulin. *The strategy of social choice*. North-Holland, 1983.

[12] R.B. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, 1991.

[13] G. Owen. *Game Theory*. Academic Press, 1982.

[14] M. Pauly. *Logic for Social Software*. PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, Amsterdam, 2001.

[15] W. van der Hoek and M. Wooldridge. On the logic of cooperation and propositional control. *Artificial Intelligence*, 164(1-2):81–119, 2005.

[16] M. Wooldridge and P. Dunne. On the computational complexity of qualitative coalitional games. *Artificial Intelligence*, 158/1:27–73, 2004.